MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. I.

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THE "EDUCATIONAL TIMES."

EDITED BY

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1288 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—Find the average of the *lengths*, and also of the *areas* of the sections, of an indefinite number of parallels of latitude on the surface of the earth; supposing these parallels to be drawn (i.) at equal intervals of *normal* latitude, (ii.) at equal intervals of *geocentric* latitude.

Solution by the PROPOSER.

Considering the earth as an *oblate spheroid*, whose *equatorial* radius is a, *polar* radius b, and eccentricity e, let L_1 be the length of a *parallel* of which λ_1 is the *true* or *normal* latitude, that is, the altitude of the elevated pole, or the inclination of a *normal* to the plane of the equator; and let A_1 be the *area* of a section of the earth through the parallel (L_1); also let L_2, A_2 be the length and area of a parallel of which λ_2 is the *geocentric* latitude, or the inclination of the *radius* to the plane of the equator.

Then we shall have

$$L_1=\frac{2\pi a\cos\lambda_1}{(1-e^2\sin^2\lambda_1)^{\frac{1}{2}}};\quad A_1=\frac{\pi a^2\cos^2\lambda_1}{1-e^2\sin^2\lambda_1};$$

$$L_2=\frac{2\pi b\cos\lambda_2}{(1-e^2\cos^2\lambda_2)^{\frac{1}{2}}};\quad A_2=\frac{\pi b^2\cos^2\lambda_2}{1-e^2\cos^2\lambda_2}.$$

Now suppose a number n $(=\frac{1}{2}\pi : \delta\lambda_1)$ of parallels to be drawn *at equal intervals* $(\delta\lambda_1)$ *of normal latitude*, and extending from the *pole* to the *equator*; then the *average* (L'_1) of their lengths will be

$$L'_1=\frac{\Sigma(L_1)}{n}=\frac{\Sigma(L_1\delta\lambda_1)}{\frac{1}{2}\pi}\ \ldots\ldots\ldots\ldots\ (\alpha).$$

By supposing the number (n) of parallels to *increase without limit*, and consequently their distance apart $(\delta\lambda_1)$ to *decrease without limit*, we shall obtain one of the required *averages*, which, from (α), will be

$$L'_1=\frac{1}{\frac{1}{2}\pi}\int\frac{2\pi a\cos\lambda_1\,d\lambda_1}{(1-e^2\sin^2\lambda_1)^{\frac{1}{2}}}=\frac{4a}{e}\sin^{-1}(e\sin\lambda)\ldots(\beta).$$

If the parallels (L_2) be drawn *at equal intervals* $(\delta\lambda_2)$ *of geocentric latitude* (λ_2), the *average* of their lengths will be

$$L'_2=\frac{1}{\frac{1}{2}\pi}\int\frac{2\pi b\cos\lambda_2\,d\lambda_2}{(1-e^2\cos^2\lambda_2)^{\frac{1}{2}}}$$

$$=\frac{4b}{e}\log\left[e\sin\lambda_2+(1-e^2\cos^2\lambda_2)^{\frac{1}{2}}\right]\ldots(\gamma).$$

The *averages* $(A'_1,\ A'_2)$ of the *areas* of the circular sections of the earth through the series of parallels $(L_1,\ L_2)$ will be

$$A'_1=\frac{1}{\frac{1}{2}\pi}\int\frac{\pi a^2\cos^2\lambda_1\,d\lambda_1}{1-e^2\sin^2\lambda_1}$$

$$=\frac{2a^2}{e^2}\left[\lambda_1+(1-e^2)^{\frac{1}{2}}\tan^{-1}\left(\frac{\cot\lambda_1}{\sqrt{1-e^2}}\right)\right]\ldots(\delta);$$

$$A'_2=\frac{1}{\frac{1}{2}\pi}\int\frac{\pi b^2\cos^2\lambda_2\,d\lambda_2}{1-e^2\cos^2\lambda_2}$$

$$=\frac{2b^2}{e^2}\left[-\lambda_2+(1-e^2)^{-\frac{1}{2}}\tan^{-1}\left(\frac{\tan\lambda_2}{\sqrt{1-e^2}}\right)\right]\ldots(\epsilon).$$

The integrals are all to be taken between the limits $(0,\ \frac{1}{2}\pi)$; hence from (β), (γ), (δ), (ϵ), we obtain

$$L'_1=\frac{4a}{e}\sin^{-1}(e);\quad L'_2=\frac{2b}{e}\log\left(\frac{1+e}{1-e}\right);$$

$$A'_1=\left(\frac{a}{a+b}\right)\pi a^2;\quad A'_2=\left(\frac{b}{a+b}\right)\pi a^2.$$

If the earth be considered a *sphere*, so that $e=0$, the *normal* and *geocentric latitudes* are *identical*; and the preceding results give $4a$ for the *average length* of parallels drawn at equal intervals of latitude, and $\frac{1}{2}\pi a^2$ for the *average of their areas*.

1373 (Proposed by T. T. Wilkinson, F.R.A.S., Burnley.)—Given a circle (C) and *any* point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that if LEM be drawn cutting the circle in L, M, we may always have $AE^2 = LE \cdot EM \pm BA \cdot AD$.

Solution by A. CAYLEY, F.R.S., F.S.A., Sadlerian Professor of Pure Mathematics in the University of Cambridge.

Consider a circle centre O and radius OA, and in relation thereto a point M either outside or inside the circle, and suppose that

$(OM)^2 - (OA)^2$, or the "squared outer potency" of M is denoted by $\square o \cdot M$; and

$(OA)^2 - (OM)^2$, or the "squared inner potency" of M is denoted by $\square i \cdot M$.

So that for an outside point, $\square o \cdot M, = -\square i \cdot M$, is the square of the tangential distance of M from the circle; and for an inside point, $\square i \cdot M, = -\square o \cdot M$, is the square of the shortest semichord through M.

Suppose now that M is a given point; the proposed question is in effect to find the locus of a point P such that

$$\pm \square o \cdot P \pm \square o \cdot M = (MP)^2;$$

but we have thus in reality *four* different questions according as the signs are assumed to be $+\,+$, $+\,-$, $-\,+$, or $-\,-$; the case $+\,+$, or when $\square o \cdot P + \square o \cdot M = (MP)^2$, is perhaps the most interesting.

Taking the radius as unity, (α, β) as the coordinates of M, and (x, y) as the coordinates of P, we have here

$$(x^2+y^2-1)+(\alpha^2+\beta^2-1)=(x-\alpha)^2+(y-\beta)^2.$$

Or what is the same thing,

$$\alpha x + \beta y - 1 = 0;$$

that is, the locus of P is a right line the polar of M in regard to the circle.

It may be remarked, that when M is an inside point, then throughout the locus P is an outside point; and replacing the negative quantity $\square o \cdot M$ by its value, $= -\square i \cdot M$, we have

$$\square o \cdot P - \square i \cdot M = (MP)^2.$$

If however M is an outside point, then in part of the locus P is an outside point, and we have

$$\square o \cdot P + \square o \cdot M = (MP)^2,$$

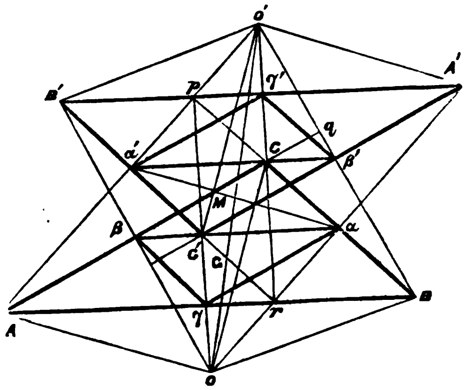
while in the remainder of the locus P is an inside point, and replacing the negative quantity $\square o \cdot P$ by its value, $= -\square i \cdot P$, we have

$$-\square i \cdot P + \square o \cdot M = (MP)^2;$$

For the case $+\,-$, the locus of P is a right line, but for each of the other two cases $-\,+$ and $-\,-$ the locus is a circle; the discussion of the several cases presents no particular difficulty.

1383 (Proposed by M. Prouhet, Paris.)—Let ABC be any triangle, α, β, γ the bisections of the sides BC, CA, AB, and O the centre of the circumscribed circle. Draw $O\alpha$, $O\beta$, $O\gamma$ and produce them to A′, B′, C′, making $OA' = 2O\alpha$, $OB' = 2O\beta$, $OC' = 2O\gamma$; also let α', β', γ' be the bisections of B′C′, C′A′, A′B′, respectively. It is required to deduce some of the leading properties of this system of conjugate triangles.

Solution by T. T. WILKINSON, F.R.A.S., Burnley.



1. Constructing the figure as in the annexed diagram, we have, obviously, $B'C' = 2\beta\gamma = BC$, $C'A' = 2\gamma\alpha = CA$, and $A'B' = 2\alpha\beta = AB$. Hence, *the triangles* ABC, A′B′C′ *have their sides parallel, and are equal in all respects, but inversely situated.*

2. Since $AO = AC' = AB'$, and α' is the *middle* of B′C′, $A\alpha'$ is *perpendicular* to B′C′; hence, if the same construction be made for the triangle A′B′C′, we shall obtain O′ for the centre of the circle about A′B′C′, and the triangle ABC will be reproduced. *The triangles* ABC, A′B′C′ *are therefore reciprocals; and so also are the triangles* $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$.

3. The lines AO′, BO′, CO′, A′O, B′O, C′O are *perpendicular* to the corresponding sides of triangles ABC, A′B′C′; hence O *and* O′ *are the intersections of the perpendiculars of the triangles* A′B′C′ *and* ABC *respectively.*

4. Join OB and O′B′; then OBO′B′ is evidently a parallelogram of which OO′ and BB′ are the diagonals. They therefore intersect in their middle point M. Similarly AOA′O′ is a parallelogram whose diagonals are OO′ and AA′. They therefore intersect in M. Lastly OCO′C′ is a parallelogram, whose diagonals OO′ and CC′ also intersect in M. Hence, *the middle* (M) of OO′ *is the centre of similitude of* ABC *and* A′B′C′.

5. Again, $A\alpha$, $B\beta$, $C\gamma$ will intersect in the *centroid* G of the triangle ABC; hence G *is the centre of similitude* of ABC *and* $\alpha\beta\gamma$.

6. Also $A'\alpha$, $B'\beta$, $C'\gamma$, pass through O; hence, O *is the centre of similitude of* A′B′C′ *and* $\alpha\beta\gamma$.

7. In like manner, O′ *is the centre of similitude of* ABC *and* $\alpha'\beta'\gamma'$.

8. Now it has been proved, (*Catalan's* "*Th. et Prob.*," p. 57, 3rd ed.), that the centres of similitude of any three polygons range in the same straight line; and hence, since O, G, M, and O, M, O′, are in a straight line, it follows that the points O, G, O′ are also in a straight line.

Hence *the centroid of a triangle, the intersection of the perpendiculars, and the centre of the circumscribed circle,* are *in the same straight line.*

9. O and O′ are *corresponding points* in the system of triangles $\alpha\beta\gamma$ and ABC, being the intersections of the perpendiculars of these triangles; but $AB = 2\alpha\beta$; and, hence *the ratio of similitude is* 1 : 2; $\therefore$ O′G=2OG.

10. Since $M\alpha = \frac{1}{2}O'A'$, $M\beta = \frac{1}{2}O'B'$, $M\gamma = \frac{1}{2}O'C'$, $\therefore M\alpha = M\beta = M\gamma$; hence M *is the centre of the circle about* $\alpha\beta\gamma$.

11. Since $M\alpha' = \frac{1}{2}OA$, $M\beta' = \frac{1}{2}OB$, $M\gamma' = \frac{1}{2}OC$, M *is also the centre of the circle about* $\alpha'\beta'\gamma'$; *and the points* α, β, γ, α', β', γ' *are in the circumference of the same circle.*

12. Let p, q, r be the feet of the perpendiculars from A, B, C, on BC, CA, AB; then since the triangle $\alpha'p\alpha$ is right-angled, and $\alpha'\alpha$ passes through M, which is its middle point, $M\alpha' = Mp = M\alpha =$ &c.

Consequently, *the middles* (α, β, γ) *of the sides of any triangle, the middles* (α', β', γ') *of the portions of the perpendiculars comprised between the point of intersection and the angles; and the feet* (p, q, r) *of the perpendiculars, are nine points in the circumference of the same circle.* It also follows that *the centre of this circle is at the middle of the line which joins the intersection of the perpendiculars with the centre of the circumscribing circle.* And further, that *the radius of this circle is half the radius of the circle about the triangle* ABC.

Scholium.—I might now proceed to show that this nine point circle (M) touches "*the inscribed, and the three escribed circles*" of the triangle ABC; and not only so, but also the whole of "*the sixteen inscribed and escribed circles*" of the four triangles ABC, AO′B, BO′C, CO′A. The *first* of these curious properties was originally given by M. Terquem, in tome i. pp. 196—8 of the *Nouvelles Annales* for 1842, and has since found its way into several elementary works; the *second* property was given, *for the first time*, by myself, as the Prize Question (1883) in the *Diary* for 1854-5, where a complete solution may be seen. Dr. Salmon, Dr. Hart, and several others, have extended M. Terquem's property to the sphere, but no one, so far as I am aware, has noticed the extension which forms the *Diary Question* just quoted, except Mr. Casey, who styles it "Sir William Hamilton's Theorem," in the *Quarterly Journal* for 1860. Taking this extension for granted, the *Nine-Point Circle* (M) is a very remarkable one with reference to the triangles ABC, A′B′C′, AO′B, A′OB′, for it is *tangential to all the inscribed and escribed circles of the system.*

[*Note.*—We add the following definitions from Catalan's "*Théorèmes et Problèmes*," p. 56, 3rd ed. "On appelle, en général, *centre de similitude* de deux lignes un point *situé de la même manière* par rapport à chacun d'elles. Le centre de similitude est *externe* lorsque les deux figures sont *directement* semblables; il est *interne* dans le cas contraire. Deux figures semblables et semblablement placées peuvent avoir à la fois deux centres de similitude. Exemple: deux circonferences. La droite qui passe par les centres de similitude de trois figures semblables et semblablement placées s'appelle *axe de similitude.*"—Ed.]

1374 (Proposed by N'Importe.)—It is affirmed by Sir W. R. Hamilton, in the Transactions of the Royal Irish Academy, that "although $e^{-\frac{1}{x^2}}$ vanishes when $x = 0$, yet x is not a factor of it." Is this strictly true?

Solution by the Proposer.

The assertion that "although $e^{-\frac{1}{x^2}}$ vanishes when $x = 0$, yet x is not a factor of it," can be justified only by imposing upon the term *factor* a restrictive meaning from which it is free in all other analytical discussions. If any algebraical expression be represented by the multiplication together of two quantities, of which one is x, then is x a multiplier of the other quantity; and, removing all restriction from the term, x may truly be regarded as a *factor* of the original expression, which expression must vanish for $x=0$, provided the other factor be either 0 or finite for that value of x, or provided that, though it be infinite, yet the product of the two factors be 0.

But it is not true that because an expression vanishes for $x=a$, that expression must *necessarily* be divisible by $x-a$, though incautious writers have sometimes asserted as much. Thus,

$$\sqrt{x}-\sqrt{a},\ \sqrt[3]{x}-\sqrt[3]{a},\ldots \sqrt[n]{x}-\sqrt[n]{a},$$

each vanish for $x = a$, and yet neither of them is divisible by $x-a$. All that is *necessary* is that the expression must be divisible either by $x-a$, or by a factor of $x-a$; that is, if not by $x-a$, then, necessarily, by either $\sqrt{x}-\sqrt{a}$, or $\sqrt{x}+\sqrt{a}$.

Taking the function proposed, we have

$$e^{-\frac{1}{x^2}} = 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{2\,.\,3x^6} + \ldots\ldots$$

$$\therefore \frac{e^{-\frac{1}{x^2}}}{x} = \frac{1}{x} - \frac{1}{x^3} + \frac{1}{2x^5} - \frac{1}{2\,.\,3x^7} + \ldots\ldots$$

$$= \left(\frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{2x^6} - \frac{1}{2\,.\,3x^8} + \ldots\ldots\right)x.$$

This is 0 for $x=0$ (See Young's Course, p. 538), and x may with propriety be called a *factor* of the expression on the right. And in every case, x may be, and is, regarded as a factor of Px, without any previous stipulation as to the constitution of P; we should use the term, quite regardless of that constitution, or in entire ignorance of it. This

we submit cannot be disputed, however limited be the sense in which the term factor is taken. Should it be said, "If x may *always* be regarded as a factor of Px, then x may be considered as a factor of *any* expression whatever; for any function F may be put into the form Px, where $P = \frac{F}{x}$," we should reply, that we are here referring to a combination of symbols of a declared form, and not to the *results* of operations which that combination would lead to. There is not the same combination of symbols in F as in $\frac{F}{x}.x$; the expressions are different in form though the same in result. Of the latter form x is a factor; but about the factors of F we know nothing. Yet if the latter expression vanish for $x=0$, we *do* come to know something about the constitution of F; we know, for instance, that F cannot be of the form $a+Qx$, and therefore that $F=Px$ cannot be of arbitrary constitution if Px vanish with x. And we think that such evanescence should justify our regarding x, or at least a factor of it, as a factor of F.

But these remarks are not intended in the slightest degree to impugn the views of the illustrious analyst alluded to in the question, but only to suggest whether the term *factor* is not sometimes used in too restrictive a sense.

1381 (Proposed by the Rev. Robert Harley, F.R.S., Brighouse, Yorkshire.) — Tschirnhausen has shown that the general equation of the fifth degree can, by the resolution of a linear and a quadratic equation, be reduced to the form

$$x^5+px^2+qx+r=0.$$

Show how to extend Tschirnhausen's method so as to reduce this equation to either of the two following forms, viz.,

$$x^5+qx+r=0,$$
$$\text{or,}\quad x^5+px^2+r=0,$$

by the aid of equations of inferior degrees.

By the EDITOR.

We have received from Mr. S. Bills a very able and elaborate investigation of this question, but we regret that our limits will not allow us to give more than the following brief sketch of it.

Eliminating x between the proposed equation

$$x^5+px^2+qx+r=0 \ldots\ldots\ldots\ldots\ldots (1)$$

and the assumed equation

$$x^4+dx^3+cx^2+bx+a=y \ldots\ldots\ldots\ldots (2)$$

there results the equation

$$y^5+Ay^4+By^3+Cy^2+Dy+E=0 \ldots\ldots (3)$$

where

$$-A=5a-3pd-4q;$$

$$+B=3pbc+4qbd+5rb+2qc^2+5rcd-3p^2c+6q^2-4pr+5pqd+3p^2d^2-12pda-16qa+10a^2;$$

$$-C=-pb^3-4qb^2c-5rb^2d+3p^2b^2+9pbca+12qbda-5rbc^2-3p^2bcd+2pqbc-5pqbd^2+15rba+(pr-8q^2)bd-(11rq+3p^3)b+6qc^2a+15rcda+p^2c^3+pqc^2d+(8rp-4q^2)c^2-9p^2ca+18q^2a+(4q^2-7pr)cd^2-(2qr-3p^3)cd-12pra+15pqda+(2p^2q+5r^2)c-(p^3-3rq)d^3+9p^2d^2a-(qp^2-5r^2)d^2-(pq^2-rp^2)d-18pda^2+p^4-4q^3+8rpq+10a^3-24qa^2;$$

while D will be of the *fourth*, and E of the *fifth* degree in a, b, c, d.

Mr. Bills exhibits the elimination in full; but as we have proposed this for a separate investigation (see Question 1401), we give here the *result* only.

In order to reduce (3) to the *first* trinomial form, A, B, and C must severally vanish. The evanescence of A gives $a=\frac{1}{5}(3pd+4q)$; and substituting this value of a in the expression for B, and equating it to zero, we have

$$5(3pc+4qd+5r)b+10qc^2+25rcd-15p^2c-3p^2d^2-23pqd-2q^2-20rp=0 \ldots\ldots\ldots\ldots (4).$$

So that, if we assume

$$3pc+4qd+5r=0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

we shall have also

$$10qc^2+25rcd-15p^2c-3p^2d^2-23pqd-2q^2-20rp=0 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

From (5) we obtain

$$c=-\frac{1}{3p}(4qd+5r);$$

and substituting this value of c in (6), the result will be a quadratic in d, where solution will give d, and thence c and a, in terms of the known coefficients p, q, and r. Substituting these values of a, c and d in the expression for C, and equating the result with zero, we obtain a cubic in b. Thus the first part of the question is solved.

For the solution of the *second* part it would be necessary to calculate the coefficient D, and to insert therein the values of a, c, and d found above; the resulting equation would be a biquadratic in b, and this function being determined, the given quintic equation will take the *second* trinomial form.

Mr. Bills calls attention to a Paper recently published by the Proposer of the Question, in the Quarterly Journal of Mathematics, (No. 21,) entitled "A Contribution to the History of the Problem of the Reduction of the general equation of the fifth degree to a trinomial form;" from which it appears that the first part of the Question was effected so early as the year 1786, by a Swedish mathematician named Erland S. Bring.

Mr. Bills adds:—"Bring's expressions for the coefficients A, B, C are the same as those given in my investigation, except the *signs* of the terms $3p^3b$, $3p^3cd$, $3rqd^3$, $5r^2d^2$, rp^2d; but I am ignorant of the method by which he obtained them. His method of effecting their evanescence differs from mine, and though ingenious, it is much less simple and direct."

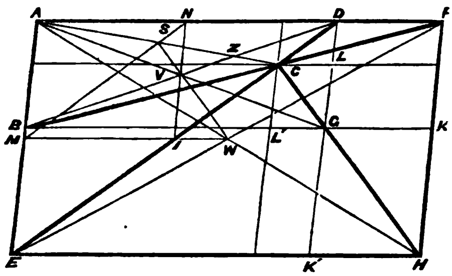
1385 (Proposed by the Rev. R. H. Wright M.A., Trinity College, Cambridge, and Head Master of the Ashford Grammar School, Kent.)

—Let ADE be a triangle, and MN a straight line terminated by AE, AD, and represented by the equation $la+m\beta+n\gamma=0$; show that the equation to the line joining A and the *middle* of MN is $(ma-lb)\,\beta+(lc-na)\,\gamma=0$. Also apply Trilinear Coordinates to prove that the middles of the three diagonals of a complete quadrilateral are in a straight line.

By the EDITOR.

The following investigation has been drawn up, with some alterations and additions, from Solutions sent by Mr. W. K. Clifford, Mr. A. Renshaw, and the Rev. R. H. Wright, M.A.

And as the Solution of Quest. 982 was published without a figure, we have drawn the annexed diagram to illustrate *both* Questions.



1. Putting a, b, c for the sides (DE, EA, AD) of the triangle of reference, the *Trilinear* equation of *any* line parallel to MN is

$$la+m\beta+n\gamma=\kappa\,(a\alpha+b\beta+c\gamma),$$

and this will pass through A if $\kappa a=l$; hence the equation of a line through A *parallel* to MN is

$$(ma-lb)\,\beta+(na-lc)\,\gamma=0 \quad \text{..........(1)}.$$

The line joining A with the *middle* of MN will be the *harmonic conjugate* of (1), relative to AD, AE; its equation will therefore be

$$(ma-lb)\,\beta+(lc-na)\,\gamma=0 \quad \text{..........(2)}.$$

2. The theorem in the *second* part of the Question is a *particular case* of the following, which we shall prove both by *Tangential* and *Trilinear Coordinates.*

A straight line (BCF) *cuts the sides* (AE, ED, DA) *of a triangle* (ADE) *in* B, C, F; *and straight lines drawn from* A, D, E, *through any point* (O) *cut the sides in* I, M, N; *then the intersections* (S) *of* MN *with* AC, (V) *of* NI *with* BD, *and* (W) *of* IM *with* EF *will be in the same straight line.*

3. To apply *Tangential Coordinates* to the theorem in Art. 2, take ADE as triangle of reference, so that the equations of the points A, D, E are $\alpha=0$, $\beta=0$, $\gamma=0$; and let the equation (3) of the point (O) and the equations (4) of the *transversal* (BCF) be

$$la+m\beta+n\gamma=0 \quad \text{....................(3)}$$

$$\lambda\alpha=\mu\beta=\nu\gamma \quad \text{......................(4)}.$$

Then the equations of the points C, M, N, will be

$$\mu\beta=\nu\gamma,\quad la+n\gamma=0,\quad la+m\beta=0;$$

hence the equation of the intersection (S) of AC, MN, will be

$$\frac{n}{\nu}\,(la+m\beta)=\frac{m}{\mu}\,(la+n\gamma).$$

The symmetry shows that the three points S, V, W lie on the *straight line*

$$\frac{l}{\lambda}\,(m\beta+n\gamma)=\frac{m}{\mu}\,(n\gamma+la)=\frac{n}{\nu}\,(la+m\beta).$$

4. If O be the *centroid* of the triangle of reference, or $l=m=n$, I, M, N will be the *middles* of its sides, and S, V, W will then be the *middles* of the three diagonals (AC, BD, EF) of the *complete quadrilateral* ABCDEF; hence these three points are in the *straight line*

$$\frac{\beta+\gamma}{\lambda}=\frac{\gamma+\alpha}{\mu}=\frac{\alpha+\beta}{\nu}.$$

5. Regarding the system of equations in Art. 3 as *Trilinear*, they give the following theorem:—

Denote the intersections of BD, EF, *of* EF, AC, *and of* AC, DB, *by* X, Y, Z; *then the three straight lines* IX, MY, NZ *will meet in a point.*

6. We may apply *Trilinear Coordinates* to prove the theorem in Art. 2, by considering (4) the equations of the point O, and (3) the equation of the transversal BCF; then the equations of the lines MN, AC; NI, BD; IM, EF; will be

$$-\lambda\alpha+\mu\beta+\nu\gamma=0,\quad m\beta+n\gamma=0;$$

$$\lambda\alpha-\mu\beta+\nu\gamma=0,\quad n\gamma+l\alpha=0;$$

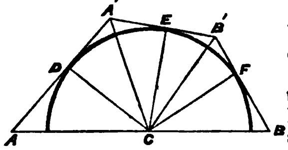
$$\lambda\alpha+\mu\beta-\nu\gamma=0,\quad l\alpha+m\beta=0:$$

and S, V, W are the intersections of these three pairs of lines; hence it is readily seen that S, V, W are in the *straight line*

$$\left(-\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}\right)\lambda\alpha+\left(\frac{\lambda}{l}-\frac{\mu}{m}+\frac{\nu}{n}\right)\mu\beta+\left(\frac{\lambda}{l}+\frac{\mu}{m}-\frac{\nu}{n}\right)\nu\gamma=0.$$

1386 (Proposed by Mr. E. B. McCormick.)—The middle point C of a straight line AB being the centre of a semi-circle of any radius, if *any* third tangent A′B′ to the semi-circle cut the tangents to it from A and B at the points A′ and B′ prove that AA′ . BB′ remains constant and equal to AC^2.

Solution by Mr. H. MURPHY; Mr. F. SINCLAIR; Mr. W. HOPPS; Mr. W. K. CLIFFORD; Mr. A. RENSHAW; R. TUCKER, M.A.; and T. T. WILKINSON, F.R.A.S.



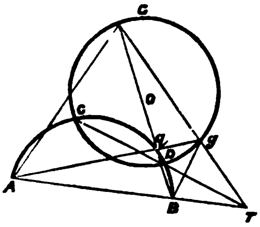
Let D, E, F, be the points of contact of AA′, A′B′, B′B; then the angles CBB′, BCF, FCB′, DCA′ are respectively equal to CAA′, ACD, ECB′, ECA′, ∴ BCB′ + DCA′ = a right angle = DCA′ + DA′C. Hence ∠ AA′C = BCB′, and the triangles ACA′, BCB′ are *similar*;

$$\therefore\ AA'\,.\,BB'=AC\,.\,CB=AC^2.$$

1388 (Proposed by T. T. Wilkinson, F.R.A.S., Burnley.)—If through the points A, B, *any* circle be drawn, cutting a given circle (O) in C, D; and if CD joined meet AB, produced, if necessary, in T; then if *any* straight line (TgG) through T meet the circle (O) in g, G, we shall always have,

$$AG \cdot GB : Ag \cdot gB = TG : Tg.$$

Solution by Mr. A. Renshaw; Mr. W. Hopps; R. Tucker, M.A.; and T. T. Wilkinson, F.R.A.S.



By a property of the circle, we have

$$GT \cdot Tg = CT \cdot TD = AT \cdot TB;$$

$$\therefore GT : TA = TB : Tg.$$

Hence (Euc. vi. 6) the triangles GTA, BTg are *similar*, and also the triangles GTB, ATg.

Therefore $AG : gB = TG : BT$,

also $GB : Ag = BT : Tg$;

therefore $AG \cdot GB : Ag \cdot gB = TG : Tg$.

Cor. 1.—In like manner it may be shown that

$$GA \cdot Ag : GB \cdot Bg = AT : BT.$$

Cor. 2.—A circle would pass through the four points A, B, g, G; hence, if AG, Bg meet in Q, and Ag, BG in q, we shall have

$$AQ \cdot QG = BQ \cdot Qg, \text{ and } Aq \cdot qg = Bq \cdot qG.$$

1389 (Proposed by * * * .)—A curve of the 3rd order, consisting of three symmetrical branches, is drawn so as to touch the sides of an equilateral triangle at their middle points. These three points are joined so as to form a new equilateral triangle. Show that if PA, PB, PC be the perpendiculars from any point P on the curve upon the sides of one equilateral triangle, and PD, PE, PF the perpendiculars from the same point on the sides of the other equilateral triangle, then the ratio

$$\text{vol. } PA \cdot PB \cdot PC : \text{vol. } PD \cdot PE \cdot PF$$

is constant, wherever P be taken on the curve.

Solution by Mr. W. K. Clifford.

The general equation to a cubic touching $B+C=0$, $C+A=0$, $A+B=0$, where they meet $A=0$, $B=0$, $C=0$, is evidently

$$ABC = \kappa\,(B+C)\,(C+A)\,(A+B) \text{......(1).}$$

In the case supposed, let $B+C \equiv \alpha$, $C+A \equiv \beta$, $A+B \quad \gamma$, then (1) becomes

$$(\alpha+\beta-\gamma)\,(\alpha-\beta+\gamma)\,(-\alpha+\beta+\gamma) = \kappa\alpha\beta\gamma,$$

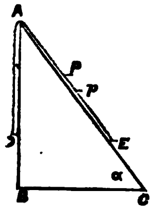
which expresses the property in question.

The asymptotes are parallel to $\alpha\beta\gamma$, and there are three points of inflexion, all at an infinite distance.

The proof holds if the cubic touch the sides of any triangle in three points, such that the lines joining them to the opposite vertices meet in a point.

1352 (Proposed by Dr. Rutherford, F.R.A.S., Royal Military Academy, Woolwich.)—An elastic string, whose weight is W, is laid over the top of an inclined plane in such a manner as to remain at rest; find how much the string is stretched, λ being the modulus of elasticity, a the natural length of the string, and α the inclination of the plane.

Solution by the Proposer.



Let $AE = a'$, the natural length of $AE = b'$;

$AD = a''$, the natural length of $AD = b''$;

$AP = x$, the natural length of $AP = x'$;

$Pp = dx$, the natural length of $Pp = dx'$;

t, $t+dt$ = the tensions at P, p;

w_1, w_2 = the weights of AE, AD.

Then $\dfrac{w_1}{b'}\,dx'$ = weight of element Pp,

hence its pressure along $AC = \dfrac{w_1 \sin\alpha}{b'}\,dx'$.

Now in the case of equilibrium of Pp we have

$$t + dt + \frac{w_1 \sin\alpha}{b'}\,dx' - t = 0,$$

$$\therefore dt = -\frac{w_1 \sin\alpha}{b'}\,dx';$$

$$\text{integrating, } t = C - \frac{w_1 \sin\alpha}{b'}\,x'.$$

When $x' = b'$, then $t = 0$; hence $C = w_1 \sin\alpha$,

$$\therefore t = w_1 \sin\alpha\left(1 - \frac{x'}{b'}\right).$$

Now, by Hooke's law,

$$dx = dx'\,(1 + \lambda t)$$

$$= dx'\left(1 + \lambda w_1 \sin\alpha - \lambda w_1 \sin\alpha\,\frac{x'}{b'}\right);$$

$$\text{integrating, } x = x' + \lambda w_1 x' \sin\alpha - \tfrac{1}{2}\lambda w_1 \sin\alpha\,\frac{x'^2}{b'}.$$

When $x = a'$, $x' = b'$, we have

$$a' - b' = \tfrac{1}{2}\lambda w_1 b' \sin\alpha \text{ .... (1).}$$

Similarly, $a'' - b'' = \tfrac{1}{2}\lambda w_2 b''$ .... (2).

But $b' + b'' = a$, $w_2 = w_1 \sin\alpha$,

$$\therefore w_2\,(1 + \sin\alpha) = W \sin\alpha;$$

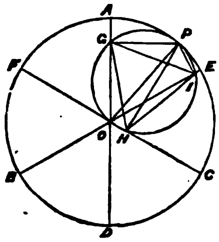
hence, from (1) + (2), we have

$$a' + a'' - a = \tfrac{1}{2}\lambda a w_2;$$

$$\therefore \text{increase of length of string} = \tfrac{1}{2}\lambda a W\,\frac{\sin\alpha}{1+\sin\alpha}.$$

1375 (Proposed by Mr. W. Hopps, Hull.)—The circumference of a circle is divided into any *odd* number of equal arcs, in any one of which a point is taken, and perpendiculars are drawn therefrom upon the diameters which pass through the points of division. Prove that the sum of the squares of the segments of these diameters is constant.

Solution by the Proposer.



In the diagram the circumference of the *greater* circle is divided into *three* equal arcs in the points A, B, C, and P is a point in the arc AC. Draw PG, PI, PH perpendicular to the diameters AD, BE, CF. Let O be the centre of the circle, and represent its radius by r; also draw OP, GH, HI, IG. Then PGO, PIO, PHO being right $\angle$s, the circle whose diameter is OP passes through P, G, O, H, I. Hence $\angle$ IGH $=$ IOH $=\frac{1}{3}$ of two right $\angle$s $=$ GHI $=$ GEH; therefore the circumference of the *less* circle is also divided into *three* equal arcs in the points G, H, I. We have, then, by Prop. iv., Stewart's Theorems,

$$2\,PG^2+2PH^2+2PI^2=3r^2 \ldots\ldots\ldots\ldots (1).$$

But $AD^2+BE^2+CF^2=AG^2+GD^2+BI^2+IE^2+CH^2+HF^2+2AG\,.\,GD+2BI\,.\,IE+2CH\,.\,HF=AG^2+GD^2+BI^2+IE^2+CH^2+HF^2+3r^2$.

$$\therefore AG^2+GD^2+BI^2+IE^2+CH^2+HF^2=9r^2.$$

Now, it will be found that whatever the *odd* number of equal arcs may be, the feet of the perpendiculars will always divide the circumference of the *less* circle into the *same odd* number of equal arcs; from which the truth of the property stated in the Question is sufficiently manifest.

Cor. 1.—From (1) it appears that the sum of the squares of the perpendiculars is *constant* also.

Cor. 2.—As the positions of the points G, H, I vary as that of P varies, we may suppose O to be *any* point in the arc GH of the *less* circle, and then by supposing PA, PB, PC to be drawn, we have $PA^2+PB^2+PC^2=2r\,(AG+BI+GH)$.

And, by the Proposition in Stewart's Theorems already cited, $PA^2+PB^2+PC^2=6r^2$;

$$\therefore AG+BI+CH=3r \ldots\ldots\ldots\ldots (2).$$

$$\text{But, } \left.\begin{aligned} AG&=OA-OG=r-OG \\ BI&=BO+OI\ =r+OI \\ CH&=OC-OH=r-OH \end{aligned}\right\} \ldots\ldots\ldots (3);$$

$$\therefore \text{ by (2), } OG+OH=OI.$$

Now it will be found that whatever odd number of equal arcs the *less* circle may be divided into, the signs $-$ and $+$ in (3) will always *alternate*. Hence we have a solution of Question 1291, "Educational Times," different from those which appeared in the number for Sept. 1862.

Cor. 3.—In (2) is expressed a property similar to that stated in Question 27 of the "Key" for March 7th, 1863.

Cor. 4.—From what precedes it is clear that when the circumference of the *larger* circle is divided into any *even* number of equal arcs, the property expressed in the question still subsists; the diameters then being those joining the opposite points of division of the circumference.

[*Note.*—G. H. S., and R. Tucker, M.A., give the following proof of the theorem.

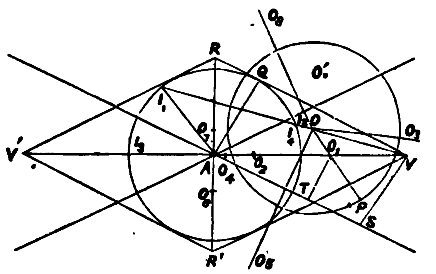
Let $\angle$ POA$=\theta$, AOB$=$BOC$=$&c.$=\alpha$, so that $(2n+1)\,\alpha=2\pi$, and S$=$the required sum; then

$$AG^2+GD^2=2OA^2+2OG^2=r^2\,(3+\cos 2\theta);$$

$$\therefore \frac{S}{r^2}=3\,(2n+1)+\cos 2\theta+\cos 2\,(\theta+\alpha)+$$
$$\cos 2\,(\theta+2\alpha)+\ldots\ldots\cos 2\,(\theta+n\alpha)$$
$$=3\,(2n+1)+\cos 2\,(\theta+n\alpha)\sin(2n+1)\,\alpha \operatorname{cosec}\alpha;$$
$$\text{but } \sin(2n+1)\,\alpha=\sin 2\pi=0,$$
$$\therefore S=3\,(2n+1)\,r^2=\text{a } constant.\text{—Ed.]}$$

1391 (Proposed by the Rev. W. Mason, Normanton, Yorkshire.)—Required the *area-locus* of the centre of a circle of given radius, tangent to which, and to two given straight lines, *eight* circles can be drawn.

Solution by the Rev. W. Mason; and Mr. Albert Escott, Royal Hospital Schools, Greenwich.



Let O be the centre of the circle; and with A, the intersection of the straight lines, as centre, describe a circle of an equal radius; draw the tangents RV, R'V', RV', R'V parallel to the given straight lines. Through O draw VOI_2I_1, $V'OI_3I_4$; then OO_1, OO_2, OO_3, OO_4, drawn parallel to AI_1, AI_2, AI_3, AI_4, will give O_1, O_2, O_3, O_4, the centres of *four* of the *eight* circles. The same process, repeated with R, R', will give O_5, O_6, O_7, O_8, the remaining four centres, on the other diagonal of the rhombus. But if O be without the rhombus, at O' for instance, O'R, O'V will not cut the circle, and there will be only *four* tangent circles.

The rhombus VRV'R' is, therefore, the *area-locus* required.

Cor.—Let r ($=$OI) be the radius of the *given* circle, a ($=$OA) the distance of its centre (O) from the intersection (A) of the given straight lines, and $\rho_1, \rho_2 \ldots \rho_8$ the radii of the *eight tangent* circles; then, if VR touch the circle A in Q, we shall have

$$OO_1 : AI_1 = VO : VI_1,$$
$$\rho_1 : r = OI_1 : VI_1;$$
$$\rho_2 : r = OI_2 : VI_2;$$
$$\rho_1\rho_2 (=\rho_3\rho_4) : r^2 = a^2 - r^2 : VQ^2;$$
$$\rho_5\rho_6 (=\rho_7\rho_8) : r^2 = a^2 - r^2 : RQ^2;$$
$$VQ \cdot QR = AQ^2 = r^2;$$
$$\therefore \rho_1\rho_2\rho_3\rho_4\rho_5\rho_6\rho_7\rho_8 = (a^2 - r^2)^4.$$

Hence if O be a *fixed* point (or limited to the *circumference* of a circle, centre A, radius a) *the continued product of the radii of the eight tangent circles is constant.* An *analytical* proof of this theorem is given on p. 52 of the *Ladies' Diary* for 1861, (Quest. 1963).

[*Note.*—That O_1, O_2, &c. *are* the centres of the tangent circles, may be shown as follows.

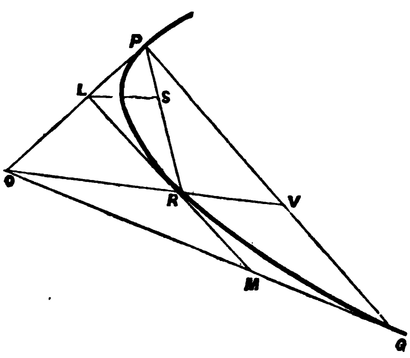
Let OO_1 meet the given circle (O) in P, and draw O_1T, VS perpendiculars to one of the given straight lines; then

$$O_1T : VS = AO_1 : AV = O_1P : AI_1;$$
$$\text{but } VS = AI_1, \therefore O_1T = O_1P.$$

If O fall within the circle AP, the *four* given lines radiating from A will be *all* cut by the circle (O), and there will then be *four internal* and *four external* tangent circles. But if O fall on the other part of the rhombus VV′, the circle (O) will cut only *two* of the four lines, and there will then be *two internal* and *six external* tangent circles. Thus, in the *first* case, the *area-locus* is the circle AP, and in the *second* case the other part of the rhombus VV′.—Ed.]

1392 (Proposed by Iota.)—Find the normal which cuts off the *least* area from a parabola.

Solution by the Proposer.



Let PQ be a normal at the point P. Draw PO, QO tangents at P, Q, and the diameter ORV. Let LRM be the tangent at R, and join PR. Draw LS perpendicular to PR.

We know that the area PRQ is two-thirds of the triangle POQ, and therefore proportional to LPR. But LPQ is a right angle, and LR parallel to PQ; therefore PLR is a right angle, L is on the directrix, and S is the focus. It follows that P must be the *end of the latus rectum*, for then SL and PR are both least.

[Mr. Escott solves the Problem *algebraically;* thus, denoting the points P, Q by (x_1, y_1), (x_2, y_2), and the parabola PRQ by $y^2 = 4ax$, the equation of the normal at P will be $2a(y - y_1) + y_1(x - x_1) = 0$, and we readily obtain

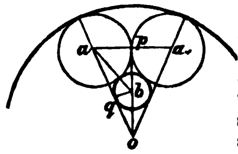
$$x_2 = \frac{(x_1 + 2a)^2}{x_1},\; y_2 = -\frac{y_1}{x_1}(x_1 + 2a); \text{ and thence}$$

$$\text{area of segment PRQ} = \tfrac{8}{3}\sqrt{a}\left(\frac{x_1 + a}{\sqrt{x_1}}\right)^3,$$

which is a *minimum* when $x_1 = a$, that is, when the normal (PQ) is drawn at the *end* (P) *of the latus rectum* of the parabola.]

1395 (Proposed by Mr. S. Bills.)—Within a given circle (radius R) let n equal circles be described, each touching the given circle and two of the series of equal circles. Let a second series of n equal circles be described, each touching two of the preceding series, and two of the new series. Let a third series of n equal circles be described in a similar manner, and so on to infinity. It is required to find the sum of the areas of all the circles thus described. Also give an example when $n = 6$.

Solution by the Proposer.



Let O be the centre of the given circle, (radius R); also let a, a_1 &c. be the centres of the *first* series of inscribed circles (radii r), and b the centre of one of the *second* series (radius r_1). Join aa_1, which will pass through the point of contact p; also join ab, and draw bq perpendicular to Oa. Let s denote the sine and c the cosine of the angle aOp $\left(=\frac{\pi}{n}\right)$; then in the triangles aOp, aOb, we have

$$ap = r,\; aO = R - r = \frac{r}{s};\; \therefore r = \frac{Rs}{1+s};$$

$$Oa = R - r,\; ab = r + r_1,\; Ob = \frac{r_1}{s};$$

$$\therefore (r + r_1)^2 = (R - r)^2 + \frac{r_1^2}{s^2} - \frac{2r_1c(R - r)}{s},$$
$$= \frac{r^2}{s^2} + \frac{r_1^2}{s^2} - \frac{2rr_1c}{s^2},$$

whence we find $r_1 = mr$, where

$$m = \frac{s^2 + c}{c^2} - \left\{\left(\frac{s^2 + c}{c^2}\right)^2 - 1\right\}^{\frac{1}{2}}.$$

Now area of circle $(a) = r^2\pi$, and area of circle $(b) = r_1^2\pi = m^2r^2\pi$. In like manner, if r_2, r_3, &c. denote the radii of the succeeding series of circles, we shall have area of circle (radius r_2) $= r_2^2\pi = m^2r_1^2\pi = m^4r^2\pi$, and so on.

Hence, if A denote the area of the entire system of circles, we shall have

$$A = nr^2\pi\,(1 + m^2 + m^4 + m^6 + \dots) = \frac{n\pi}{1-m^2}\left(\frac{Rs}{1+s}\right)^2.$$

If $n = 6$, then $s = \frac{1}{2}$, $c = \frac{1}{2}\sqrt{3}$, $m = \cdot 3861062$; whence $A = (\cdot 783466)\,C$, where $C = R^2\pi$ = area of given circle.

[*Note.*—If we put $\dfrac{s^2+c}{c^2} = \sec\theta$,

we shall have

$$m = \sec\theta - \tan\theta = \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta\right),$$

$$\text{and } A = \frac{1+\operatorname{cosec}\theta}{\left(1+\operatorname{cosec}\frac{\pi}{n}\right)^2}\left(\frac{nC}{2}\right).$$

When $n=6$, we have

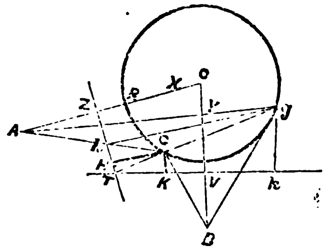
$$m = \tfrac{1}{3}\left\{1 + 2\sqrt{3} - 2\sqrt{(1+\sqrt{3})}\right\},$$

$$\frac{A}{C} = \frac{1}{3} + \frac{1+2\sqrt{3}}{6\sqrt{(1+\sqrt{3})}}.\text{—Ed.]}$$

1396 (Proposed by T. T. Wilkinson, F.R.A.S., Burnley.)—A circle (C) and two points (A, B) being given in magnitude and position, two lines ZT, VT, and a third point T, may be found, such that if from T *any* straight line TGg be drawn, cutting the circle in G and g, we shall always have

AG . GB : Ag . gB = GT : gT.

Solution by T. T. Wilkinson, F.R.A.S., &c., Grammar School, Burnley.



Construction.—Let C be the centre of the given circle; A, B, the given points; draw AC, BC, and in them take the points X, Y, such that AC . CX = BC . CY = CR^2, where CR is the radius of (C). Bisect AX, BY, in Z and V; draw ZT, VT, respectively perpendicular to AC, BC, to intersect in T; then T is the *point*, and ZT, VT, are the *lines*, required to be found in the *porism*.

Demonstration.—Let TGg be *any* straight line cutting (C) in G and g; draw GH, gh, perpendicular to ZT, and GK, gk, perpendicular to VT; also join AG, BG, Ag, Bg.

Then by Stewart's *General Theorems*, Prop. vi.,

$AG^2 : Ag^2 = GH : gh = GT : gT = BG^2 : Bg^2$;

$\therefore$ AG : Ag = BG : Bg, and

AG . GB : Ag . gB = $BG^2 : Bg^2$ = GT : gT.

But TGg is *any* straight line, and hence the property is universally true.

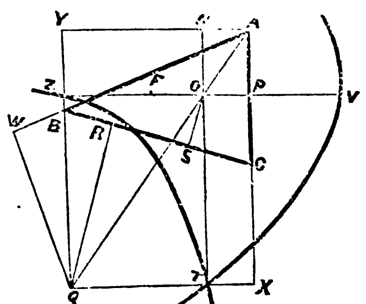
Scholium.—The principal steps of the solutions to Questions 1388 and 1396 were contained on a loose slip of paper, sent me by Mr. Miller, which appears to have been written by the Rev. Mark Noble, a learned antiquary and geometer, while on a visit at Chatham House, Ramsgate. The scrap is endorsed as "*communicated* [*to him*] *by* [*the late Sir*] *James Ivory, December* 16*th*, 1812."

The properties appear to have been intended by their author as a portion of the series of "Geometrical Propositions" commenced by him in Vol. I., Part ii., pp. 17—21, of the New Series of Leybourn's *Mathematical Repository*, but which he did not continue beyond Stewart's first Porism and his own generalization of it in Prop. iii. The remainder of Dr. Stewart's properties were demonstrated *analytically*, by Professor Galloway, in Vol. V., Part ii., pp. 76—111, of the same series of the *Repository*, to which the reader may be referred for further information.

Dr. Stewart's sixth proposition, above referred to, is as follows. "Let A, X, be *any* two points in the semi-diameter of circle (C), and let AC . CX = CR^2; bisect AX in Z, and draw ZT perpendicular to AC; from A draw *any* line AG, to any point G in the circle, also draw GH perpendicular to ZT; then AG^2 = 2GH . AC."

1398 (Proposed by Mr. W. H. Levy.)—Given the area, the difference of the sides, and the radius of the escribed circle touching the base and the sides produced; to *construct* the triangle.

Solution by Mr. W. H. Levy; and Alpha.



Construction.— Construct a rectangle APZY equal to the *given area*, having a side AY equal to the given radius; in ZP produced find a point V such that (D being the *given difference of the sides*) 4ZP . PV = D^2; and take VF = PZ. With V as vertex, and F as focus, construct a *parabola* VT; again, with AX, AY as asymptotes, draw

the *rectangular hyperbola* ZT through the point Z; and through T, the point of intersection of these curves, draw TON, XTQ parallel to AX, AY, thus forming the *rectangles* AXQY, PXTO From O, Q as centres, with OP, QX as radii, describe circles (O), (Q); from A draw ABW touching the circle Q in W; and draw a common tangent BRSC to the circles (O), (Q), touching them in S, R, and meeting AW, AX in B, C; then ABC will be the triangle required.

Demonstration.—By a property of the *hyperbola*, AT = AZ, and OX = OY; hence AOQ is a *straight line* which bisects the angle BAC, and (O), (Q) are the *inscribed* and one of the *escribed* circles of the triangle ABC. We have, therefore,

$$\triangle\ ABC = OP \,.\, AX = AZ = \textit{the given area};$$
$$\text{and } QX = AY = \text{the given radius}.$$

Also it may be easily shown that $RS = AB - AC$, $BR = CS = CP$, $BC = PX = OT$, $BS \,.\, SC = OP \,.\, QX$, and $RS^2 = BC^2 - 4BS \,.\, SC$; hence, by a property of the *parabola*, $D^2 = 4PZ \,.\, PV = 4VF \,.\, VO - 4PZ \,.\, PO = OT^2 - 4OP \,.\, QX = BC^2 - 4BS \,.\, SC = RS^2$; $\therefore D = RS = AB - AC$.

ABC is, therefore, the triangle required; since its area is equal to the *given area*, the radius of the circle escribed to BC is equal to the *given radius*, and the difference of its sides is equal to the *given difference*.

1399 (Proposed by Mr. W. K. Clifford.)—From a point A two chords are drawn meeting a conic section in four points B, joined also by four straight lines a. These intersect two and two in two points P lying on the Polar of A. At the points B are drawn four tangents b, which intersect in six points, two of which are on the polar of A, and the others lie two and two on the two straight lines AP. These tangents intersect the original chords in 4 points, which may be joined by 4 straight lines intersecting by pairs in the points P. The lines a and b intersect in 8 points C, which may be joined by 20 lines c; 4 of these pass through A, and the others may be divided into groups of 4. Each group has 6 intersections, two of which lie on the polar of A, and the others lie two and two on lines through A. Any two groups intersect in 8 points, having properties like those of the points C.

Solution by the Proposer.

Take the chords through A for the sides β, γ of the triangle of reference, one of the straight lines a as the side α; and let another of them be represented by

$$l\alpha + m\beta + n\gamma\ (=\delta) = 0.$$

Let the equation to the conic be

$$\alpha\delta = k\beta\gamma \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

then the four lines a are

$$\left.\begin{array}{l|l} a_1 \ldots\ldots\ldots\ldots \alpha = 0 & a_3 \ldots l\alpha + m\beta = 0 \\ a_2 \ldots\ldots\ldots\ldots \delta = 0 & a_4 \ldots l\alpha + n\gamma = 0 \end{array}\right\} \ldots(2),$$

The polar of A is evidently

$$2l\alpha + m\beta + n\gamma = 0 \qquad\ldots\ldots\ldots\ldots\ldots(3).$$

The equations to the tangents b are

$$\left.\begin{array}{l|l} b_1 \ldots m\alpha - k\gamma = 0 & b_3 \ldots m\delta + lk\gamma = 0 \\ b_2 \ldots n\alpha - k\beta = 0 & b_4 \ldots n\delta + lk\beta = 0 \end{array}\right\} \ldots(4).$$

For instance, assume $\delta = p\gamma$ for the equation to b_3; then from (1) we find that this meets the curve where it meets $p\alpha = k\beta$; but as b_3 is a *tangent*, this must coincide with a_3, or $mp = -lk$; and thus b_3 becomes $m\delta + lk\gamma = 0$.

The intersections of b_1b_2 and of b_3b_4 lie on $m\beta - n\gamma = 0$; of b_1b_4 and of b_2b_3 on $m\beta + n\gamma = 0$; and of b_1b_3 and b_2b_4 on $2l\alpha + m\beta + n\gamma = 0$.

The equations of the four lines joining intersections of the lines b with β and γ are

$$\left.\begin{array}{l} mn\delta + lk\,(m\beta + n\gamma) = 0 \\ mn\alpha - k\,(m\beta + n\gamma) = 0 \\ mn\,(l\alpha + n\gamma) - lk\,(m\beta - n\gamma) = 0 \\ mn\,(l\alpha + m\beta) + lk\,(m\beta - n\gamma) = 0 \end{array}\right\} \ldots\ldots(5).$$

The first pair meet $2l\alpha + m\beta + n\gamma = 0$ where it meets $m\beta + n\gamma = 0$; the second pair where it meets $m\beta - n\gamma = 0$.

The eight points C may be represented as follows:—

$$\begin{array}{cccccccc} a_4b_3, & a_3b_4, & a_2b_2, & a_1b_3, & a_3b_1, & a_4b_2, & a_1b_4, & a_2b_1. \\ E & F & G & H & K & L & M & N \end{array}$$

With this notation, the five groups of lines C are

(EK, FL, GM, HN); (EF, GH, KL, MN); (EG, HL, KM, NF); (EM, FH, NL, GK); (EN, FG, ML, KH).

All the lines of the first group pass through A, and have for equations

$$\left.\begin{array}{l|l} EK,\ldots m^2\beta + kl\gamma = 0 & GM,\ldots n^2\gamma + (mn + kl)\,\beta = 0 \\ FL,\ldots n^2\gamma + kl\beta = 0 & HN,\ldots m^2\beta + (mn + kl)\gamma = 0 \end{array}\right\} \ldots\ldots(6),$$

which may be easily verified.

As an example of the others, take the equations of the lines of the fourth groups:—

$$\left.\begin{array}{l} EM,\ldots (l^2k^2 + 2klmn)\,(l\alpha + n\gamma) = (lk + mn)\,(m\delta + kl\gamma)\,n \\ FH,\ldots (l^2k^2 + 2klmn)\,(l\alpha + m\beta) = (lk + mn)\,(n\delta + kl\beta)m \\ NL,\ldots k\,(mn + kl)\,\delta = mn^2\,(m\alpha - k\gamma) \\ GK,\ldots k\,(mn + kl)\,\delta = m^2n\,(n\alpha - k\beta) \end{array}\right\} \ldots(7)$$

The intersections of EM and FH, and of NL and GK, lie on $m\beta - n\gamma = 0$; those of EM and NL, and of FH and GK, on $m\beta + n\gamma = 0$; those of EM and GK, and of NL and FH, on

$$2l\alpha + m\beta + n\gamma = 0;$$

and so with each of the other groups.

I was wrong in saying that *any* two groups intersect in eight points, &c.; this is true of the last four, for it will be found that any two of these form two quadrilaterals, the vertices of one resting on the sides of the other, two diagonals of each passing through A, and the others being identical ($2l\alpha + m\beta + n\gamma = 0$); hence, by the converse of the first part of the question, a conic may be inscribed in one so as to circumscribe the other, and the preceding reasoning applies. But the first group is an exception; there are only four new points found by combining it with

any of the others, and these may be joined by four lines meeting two and two on the polar of A.

All these theorems may be proved very readily by projecting the conic into a circle whose centre is the projection of A.

1400 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—Apply the method of *Tangential Co-ordinates* to prove that the centroid of a triangle, the intersection of the perpendiculars, and the centre of the circumscribed circle, are in the same straight line.

Solution by the Proposer.

If ABC be the triangle of reference, and O the centre of the circle described about it, the *tangential equations* of the diameter (AOD) through A will be

$$\alpha=0,\ \frac{-\beta}{\gamma}=\frac{\sin \mathrm{BOD}}{\sin \mathrm{COD}}=\frac{\sin 2\mathrm{BCD}}{\sin 2\mathrm{CBD}}=\frac{\sin 2\mathrm{ACB}}{\sin 2\mathrm{ABC}},$$

$$\text{or } \alpha=0,\ \beta \sin 2\mathrm{B}+\gamma \sin 2\mathrm{C}=0.$$

Similarly, the equations of diameters through B, C are

$$\beta=0,\ \alpha \sin 2\mathrm{A}+\gamma \sin 2\mathrm{C}=0;$$

$$\gamma=0,\ \alpha \sin 2\mathrm{A}+\beta \sin 2\mathrm{B}=0;$$

hence the *tangential equation* of the *centre* (O) of *the circumscribed circle* is

$$\alpha \sin 2\mathrm{A}+\beta \sin 2\mathrm{B}+\gamma \sin 2\mathrm{C}=0 \ \ldots\ldots(1).$$

In like manner it may be shown that the equation of the *intersection of the perpendiculars* is

$$\alpha \tan \mathrm{A}+\beta \tan \mathrm{B}+\gamma \tan \mathrm{C}=0 \ \ldots\ldots(2),$$

and that of the *centroid*

$$\alpha+\beta+\gamma=0 \ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

These three points (1), (2), (3) will be in *the same straight line* if the determinant

$$\begin{vmatrix} \sin 2\mathrm{A}, & \sin 2\mathrm{B}, & \sin 2\mathrm{C} \\ \tan \mathrm{A}, & \tan \mathrm{B}, & \tan \mathrm{C} \\ 1, & 1, & 1 \end{vmatrix}$$

is *zero*; and it is easily seen, by expansion, that the determinant *vanishes identically*, which proves the theorem in the Question.

1366 (Proposed by Matthew Collins, B.A., Senior Moderator in Mathematics and Physics, Trin. Coll., Dublin.)—If a, b be the major and minor semi-axes of an ellipse, and R that *radius of curvature* which is, both in magnitude and position, a *chord* of the ellipse, prove that $\frac{2}{3}(\mathrm{R}ab)^{\frac{2}{3}}$ is an harmonic mean between a^2 and b^2, and that the inclination of R to the minor axis is $\frac{1}{2}\cos^{-1}\dfrac{a^2+b^2}{3(a^2-b^2)}$.

Solution by Matthew Collins, B.A.; and Alpha.

Taking the principal axes of the ellipse as axes of reference; let (ξ, η) be the centre of curvature of the point (x, y) of the ellipse; then, putting $a^2-b^2=c^2$, we shall have, when the point (ξ, η) is on the ellipse,

$$a^2\eta^2+b^2\xi^2=a^2y^2+b^2x^2=a^2b^2 \ \ldots\ldots\ldots\ldots(1)$$

$$a^4\xi=c^2x^3,\ b^4\eta=-c^2y^3 \ \ldots\ldots\ldots\ldots\ldots(2)$$

$$\mathrm{R}^2=(x-\xi)^2+(y-\eta)^2 \ \ldots\ldots\ldots\ldots\ldots(3).$$

Eliminating ξ, η, y from (1), (2), the result is

$$\left(x^2-\frac{a^4}{c^2}\right)^2\left\{x^2-\frac{a^4(a^2-2b^2)}{c^2(a^2+b^2)}\right\}=0 \ \ldots(4).$$

The *first* factor of (4) is inadmissible, since it gives $x>a$; but from the *second* we have

$$x=+\frac{a^2(a^2-2b^2)^{\frac{1}{2}}}{c(a^2+b^2)^{\frac{1}{2}}};\ y=\pm\frac{b^2(2a^2-b^2)^{\frac{1}{2}}}{c(a^2+b^2)^{\frac{1}{2}}} \ \ldots(5);$$

and then by (2)

$$\xi=\pm\frac{a^2(a^2-2b^2)^{\frac{3}{2}}}{c(a^2+b^2)^{\frac{3}{2}}};\ \eta=\mp\frac{b^2(2a^2-b^2)^{\frac{3}{2}}}{c(a^2+b^2)^{\frac{3}{2}}} \ \ldots(6).$$

From (3), (5), (6), we have $\frac{2}{3}(\mathrm{R}ab)^{\frac{2}{3}}=\dfrac{2a^2b^2}{a^2+b^2}$

$=$ the harmonic mean between a^2 and b^2.

Again, if ϕ be the inclination of R to the minor axis, we have

$$\tan^2\phi=\left(\frac{x-\xi}{y-\eta}\right)^2=\frac{a^2-2b^2}{2a^2-b^2};$$

$$\therefore \cos 2\phi=\frac{1-\tan^2\phi}{1+\tan^2\phi}=\frac{a^2+b^2}{3(a^2-b^2)}$$

Cor. 1.—From (2), we have

$$x=a^{\frac{4}{3}}c^{-\frac{2}{3}}\xi^{\frac{1}{3}},\ y=-b^{\frac{4}{3}}c^{-\frac{2}{3}}\eta^{\frac{1}{3}};$$

and substituting these values in (1), we get

$$(a\xi)^{\frac{2}{3}}+(b\eta)^{\frac{2}{3}}=c^{\frac{4}{3}},$$

which is the equation to the *evolute* of the ellipse.

Cor. 2.—Equations (6) give the coordinates of the points of intersection of the *ellipse* and its *evolute*; and from them we infer that the ellipse does not meet its evolute when $a^2<2b^2$, and that they just meet, *without intersection*, at the ends of the minor axis, when $a^2=2b^2$.

Cor. 3.—Since the foregoing values of ξ and η necessarily satisfy the condition $a^2\eta^2+b^2\xi^2=a^2b^2$, it follows that

$$a(a-2b)^3-b(b-2a)^3=(a-b)(a+b)^3,$$

$$\text{or } (a+2b)a^3-(b+2a)b^3=(a-b)(a+b)^3;$$

which leads to a remarkable property of cube numbers; and it may be easily verified *directly*.

Cor. 4.—If the ellipse $a^2y^2+b^2x^2=a^2b^2$ pass through the centre of a circle which osculates it at the point (x, y), then

$$\frac{x^6}{a^{10}}+\frac{y^6}{b^{10}}=\frac{1}{c^4}.$$

Cor. 5.—From (5) and (6), we obtain

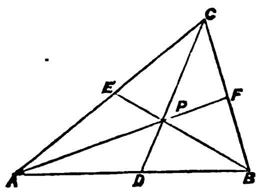
$$\frac{\xi}{x}+\frac{\eta}{y}+1=0.$$

When, therefore, the point of osculation (x, y) is given, the locus of the centre of curvature (ξ, η), lying upon the variable ellipse whose axes are given in position, is a *straight line*; but when

the centre (ξ, η) is given, the locus of the point of osculation (x, y) is an *equilateral hyperbola.* That is, when the axes of a variable ellipse are given in position, and also a point Q on the curve at which it osculates a circle and passes through its centre P, the system of circles (P) will be *coaxial;* but when the position of the axes and a point P on a variable ellipse are given, the locus of its point of osculation with a circle whose centre is P is an *equilateral hyperbola* passing through the given centre O of the ellipse, having its own centre at P′, OP′ being equal and opposite to OP, and having its asymptotes parallel to the axes of the ellipse.

1380 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.) — A particle is kept in equilibrium by three equal attracting bodies, of which two are fixed, and the third moves in a given curve; required the locus of the attracted particle, supposing the attraction to be directly proportional to the distance.

Solution by Mr. W. K. Clifford; and Mr. F. Sinclair.



Suppose A, B to be the fixed points, C the moveable one; P any position of the particle. Then since there is equilibrium,

PA : PB : PC = sin CPB : sin APC : sin BPA.

Produce CP to meet AB in D; then

PA : PB = sin BPD : sin APD,

or, PA . PD sin APD = PB . PD sin BPD.

Hence the triangles APD, BPD are equal, and AD = DB. Similarly for the points E, F; thus DP = $\frac{1}{3}$DC; but D is a fixed point; hence P describes a curve similar to the one which C describes, having D for a centre of similitude.

[Otherwise:—By the parallelogram of forces, the resultant of the two PA, PB will be represented, both in direction and magnitude, by the double of the line (PD) joining P with the middle (D) of AB; hence PC must be in the same straight line with PD, and equal to 2PD; or DPC is a straight line, and DP = $\frac{1}{3}$DC.—Ed.]

1382 (Proposed by Mr. S. Bills.)—Divide a given number (a) into n parts, such that, if the square of one of them be either *increased* or *diminished* by m times the product of all the rest, the result shall, in *both* cases, be a rational square.

Solution by the Proposer.

Let $n-4$ of the parts be taken at pleasure, and let S denote their sum and P their product; also let px; qx, rx and $a-\mathrm{S}-(p+q+r)\,x$ denote the four remaining parts.

By the question we must have

$$p^2+mqr\mathrm{P}\,[a-\mathrm{S}-(p+q\div r)\,x]=f^2 \;\ldots\ldots(1)$$

$$p^2-mqr\mathrm{P}\,[a-\mathrm{S}-(p+q+r)\,x]=g^2 \;\ldots\ldots(2).$$

From (1) + (2) and (1) − (2), we have

$$2p^2=f^2+g^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$2mqr\mathrm{P}\,[a-\mathrm{S}-(p+q+r)\,x]=f^2-g^2 \ldots\ldots(4).$$

Now (3) is satisfied by assuming $p=h^2+k^2$, $f=2hk+(h^2-k^2)$, $g=2hk=(h^2-k^2)$; then from (4) we obtain

$$x=\frac{2mqr\mathrm{P}\,(a-\mathrm{S})-(f^2-g^2)}{2mqr\mathrm{P}\,(p+q+r)} \;\ldots\ldots\ldots\ldots(5).$$

Thus the problem is solved in all its generality.

As an example taken at random, let $a=34$, and let the number of parts into which it is to be divided be 8, and $m=5$. Take $h=2$ and $k=1$; then $p=5$, $f=7$, and $g=1$. Again; since $n-4$ of the parts are to be taken at pleasure, let those 4 parts be 1, 2, 3, 4; then will S = 10 and P = 24; and if we take $q=3$ and $r=1$, (5) will give $x=\frac{359}{135}$, and the 8 parts will be $\frac{359}{27}$, $\frac{359}{45}$, $\frac{359}{135}$, $\frac{1}{15}$, 1, 2, 3, 4, which fulfil the conditions of the question; px, or $\frac{359}{27}$, being the part to be squared.

Numerous other answers may be readily found.

1384 (Proposed by R. Tucker, M.A.)—Find the "*Radial Curves*" corresponding to the Conic Sections, the Catenary, the Semi-cubical Parabola, and the Lemniscate; and conversely, find the curves whose "*Radial Curves*" are $r\cos\theta=a$, $r\tan\theta=a$.

[*Definition.*—From a fixed point lines are drawn *equal* and *parallel* to the *Radii of Curvature* at successive points of a given curve; the extremities of these lines will trace out a curve which we propose to call the "*Radial Curve*" corresponding to the given curve.]

1397 (Proposed by R. Tucker, M.A.)—If the "*Intrinsic equation*" to a curve be known, show how to find the equation to the "*Radial curve.*" Hence show that the Radial curve for an equiangular spiral is an equiangular spiral. Find the Radial curve for the Logarithmic curve and the Cycloid, and the curve whose Radial curve is the Parabola $y^2=4ax$.

Solution by the Proposer.

1. Let (r', θ') be any point on the *Radial Curve*, and $y=f(x)$ the equation of the given curve, of which ρ is the *Radius of Curvature* corresponding to (r', θ'); then if θ' be estimated from the *negative* side of the axis of x, it will be seen, from the *Definition*, that the equations which determine the *Radial Curve* are

$$r'=\rho,\; \cot\theta'=\frac{dy}{dx},\; y=f(x) \;\ldots\ldots\ldots\ldots(\alpha).$$

$$2.\ \text{Now } \rho=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\left(\frac{d^2y}{dx^2}\right)^{-1},$$

$$\therefore\; r'\sin^3\theta'=\left(\frac{d^2y}{dx^2}\right)^{-1} \;\ldots\ldots\ldots\ldots(\beta).$$

3. Also if ψ be the angle which the tangent to the given curve makes with the axis of x, we have (Todhunter's *Diff. Calc.*, Art. 324),

$$\psi = \tan^{-1}\frac{dy}{dx} = \tfrac{1}{2}\pi - \theta',$$

$$\rho = \frac{ds}{d\psi},\ r' = -\frac{ds}{d\theta'},$$

$$\therefore\ r'd\theta' = -ds \ldots\ldots\ldots\ldots (\gamma).$$

4. If ϕ' be the angle between the radius vector and tangent at (r', θ'), we have

$$\tan\phi' = r'\frac{d\theta'}{dr'} = -\frac{ds}{dr'} = -\frac{ds}{d\rho} \ldots\ldots\ldots (\delta).$$

5. The *Radius of Curvature* of the corresponding point of the *Evolute* of the given curve is, by Art. 3,

$$\frac{d}{d\theta'}\text{(arc of Evolute)} = -\frac{d\rho}{d\theta'} = -\frac{dr'}{d\theta'} = \rho\frac{d\rho}{ds} \ldots (\epsilon).$$

6. The *Area* (A) of a portion of the *Radial Curve*, between proper limits, may be found from

$$2\text{A} = \int r'^2 d\theta' = -\int \rho\, ds = -\int \rho\frac{ds}{dx}\,dx$$

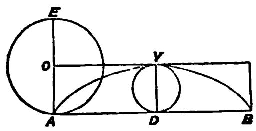
$$= -\int\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^2 \left(\frac{d^2y}{dx^2}\right)^{-1} dx \ \ldots\ldots (\zeta).$$

7. To find the *Radial Curve* for the *Cycloid* (AVB), take A as origin, and the base (AB) as axis of x; then we have

$$r'^2 = 8ay,\ \cot^2\theta' = \frac{2a-y}{y};$$

hence the equation to the *Radial Curve* is

$$r' = 4a\sin\theta',$$



which designates a *circle* whose diameter (AE) is twice that (VD) of the generating circle of the Cycloid.

Hence the *Normal* at any point of a Cycloid, being the chord of the generating circle at the corresponding point, is half the *Radius of Curvature* at the point.

8. As an application of (ζ) take the *Radial Curve* for the Cycloid, referred to the same origin as in Art. 7; then

$$\left(\frac{dy}{dx}\right)^2 = \frac{2a-y}{y},\ \frac{d^2y}{dx^2} = -\frac{a}{y^2};$$

$$\therefore\ \text{whole area} = \int_0^{a\pi} 4a\,dx = 4\pi a^2,$$

which agrees with Art. 7.

9. Our equation (γ) obviously suggests that some close relation exists between the equation to the *Radial Curve* and the *Intrinsic equation* to the curve; and on a close examination we shall find that, in general, if the *Intrinsic equation* be known, the *Radial Curve* may be easily obtained, and *vice versâ*.

Referring to Todhunter's *Integral Calculus*, Arts. 103—119, or Boole's *Differential Equations*, pp. 258—264, remembering that their ϕ is the same as our θ', we have, for the *Intrinsic equation* to the Cycloid,

$$s = 4a\sin\theta',$$

hence, by (γ), the *Radial Curve* is the *circle*

$$r' = -4a\cos\theta',$$

which agrees with Art. 7, since our origin here is the vertex (V) of the curve.

10. To find the *Radial Curve* for the *Parabola* $y^2 = 4ax$, take the vertex (A) as the fixed point; then

$$r'^2 = \frac{4}{a}(a+x)^3,\ \cot^2\theta' = \frac{a}{x};$$

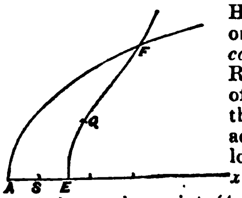
hence suppressing the *accents* (both here and similarly in the following Articles) the equation to the *Radial Curve* is

$$r\cos^3\theta = 2a,\ \text{or}\ x^3 = 2a(x^2+y^2);$$

$$\therefore\ y^2 = \frac{x^2}{2a}(x-2a),$$

$$2(2a)^{\frac{1}{2}}\frac{dy}{dx} = \frac{3x-4a}{(x-2a)^{\frac{1}{2}}},$$

$$2(2a)^{\frac{1}{2}}\frac{d^2y}{dx^2} = \frac{3x-8a}{(x-2a)^{\frac{3}{2}}}.$$



Hence we see that the origin (or vertex) is a *conjugate point* of the Radial Curve, which is of the form EQF in the figure, with an exactly similar branch below the axis of x (ASE); it cuts the parabola at the point $(4a, 4a)$, or F, and the tangent at F passes through E, making an angle $\tan^{-1} 2$ with the axis; also, since $y\dfrac{d^2y}{dx^2}$ is *negative* from $x=0$ to $x=\frac{8}{3}a$, but *positive* when x is greater than $\frac{8}{3}a$, the curve is *concave* towards the axis up to Q (where $x=\frac{8}{3}a$), and *convex* beyond that point.

11. For the *Ellipse* $a^2y^2 + b^2x^2 = a^2b^2$, take the centre as the fixed point; then, putting ϕ for the eccentric angle of a point in the curve of the ellipse, we have

$$r'^2 = \frac{a^4}{b^2}(1 - e^2\cos^2\phi)^3,\ \cot\theta' = -\frac{b}{a}\cot\phi;$$

hence the equation to the *Radial Curve* is

$$r^2(a^2\cos^2\theta + b^2\sin^2\theta)^3 = a^4b^4,\ \text{or}$$

$$(a^2x^2 + b^2y^2)^3 = a^4b^4(x^2+y^2)^2.$$

12. For the *Hyperbola*, writing $-b^2$ for b^2, the *Radial Curve* is

$$(a^2x^2 - b^2y^2)^3 = a^4b^4(x^2+y^2)^2;$$

and if $a=b$, it becomes

$$(x^2-y^2)^3 = a^2(x^2+y^2)^2,\ \text{or}$$

$$r^2\cos^3 2\theta = a^2.$$

13. For the *Catenary* $y = \frac{1}{2}c\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$,

$$r' = \frac{y^2}{c},\ \cot\theta' = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Hence the equation to the *Radial Curve* is

$$r\sin^2\theta = c, \text{ or } x^2 = \frac{y^2}{c^2}(y^2 - c^2).$$

14. For the *Logarithmic Curve* $y = ae^{\frac{x}{a}}$, we have

$$\cot\theta' = \frac{dy}{dx} = \frac{y}{a},\ \frac{d^2y}{dx^2} = \frac{y}{a^2};$$

$$\therefore \text{ by } (\beta),\ r'\sin^3\theta' = \frac{a^2}{y} = a\tan\theta';$$

hence the equation to the *Radial Curve* is

$$r\sin^2\theta\cos\theta = a, \text{ or } y^2 = \frac{ax^2}{x-a}.$$

This curve has been traced in the Solution of Question 1338, at p. 260 of the *Educational Times* for February.

15. For the *Semi-cubical Parabola* $ay^2 = x^3$,

$$\frac{dy}{dx} = \cot\theta' = \frac{3x^{\frac{1}{2}}}{2a^{\frac{1}{2}}},\ \frac{d^2y}{dx^2} = \frac{3}{4(ax)^{\frac{1}{2}}};$$

$$\therefore \text{ by } (\beta),\ r'\sin^3\theta' = \frac{4}{3}(ax)^{\frac{1}{2}} = \frac{8}{9}a\cot\theta';$$

hence the equation to the *Radial Curve* is

$$r\sin^4\theta = \frac{8}{9}a\cos\theta, \text{ or } y^4 = \frac{8}{9}ax(x^2+y^2).$$

16. For the *Lemniscate* $r^2 = a^2\cos 2\theta$,

$$\theta' = 3\theta,\ r' = \frac{a^2}{3r};$$

hence the equation to the *Radial Curve is*

$$r^2\cos\frac{2}{3}\theta = \frac{1}{3}a^2.$$

17. From (γ) it follows that if $s = f(\phi)$ be the *Instrinsic equation* to a curve, the *Radial Curve* will be $r = f'(\theta)$, the deviation ϕ being estimated from the *tangent*, and $\theta\ (= \phi)$ from the *normal* at the origin; and if $r = F(\theta)$ be the *Radial Curve*, the *Intrinsic equation* will be

$$\frac{ds}{d\phi} = F(\phi), \text{ or } s = \int F(\phi)\,d\phi.$$

Thus from Arts. 10, 13, 15, we find that the *Intrinsic equation* of the *Parabola* is

$$\frac{ds}{d\phi} = \frac{2a}{\cos^3\phi}, \text{ or } \frac{s}{a} = \tan\phi\sec\phi + \log(\tan\phi + \sec\phi);$$

that of the *Catenary*

$$\frac{ds}{d\phi} = \frac{c}{\cos^2\phi}, \text{ or } s = c\tan\phi;$$

and that of the *Semi-cubical Parabola*

$$\frac{ds}{d\phi} = \frac{8a}{9}\,\frac{\sin\phi}{\cos^4\phi}, \text{ or } s = \frac{8a}{27}(\sec^3\phi - 1),$$

s being estimated from the origin, where $\phi = 0$.

Again, since the *Intrinsic equation* $s = l\sin n\phi$ represents an epicycloid or hypocycloid according as n is less or greater than unity (Todhunter's Int. Calc., Art. 112), therefore the *Radial Curve* is

$$r = k\cos n\theta.$$

And the Intrinsic equation of the *Logarithmic spiral* being $s = ae^{k\theta}$ (Todhunter's Int. Calc., Art. 113), the *Radial Curve* is $r = ce^{k\theta}$, a similar spiral.

18. Let ρ' be the *Radius of Curvature* at right angles to ρ; then we have,—
in the *Parabola*

$$(\rho)^{-\frac{2}{3}} + (\rho')^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}};$$

in the *Ellipse*

$$(\rho)^{-\frac{2}{3}} + (\rho')^{-\frac{2}{3}} = \left(\frac{a^2}{b}\right)^{-\frac{2}{3}} + \left(\frac{b^2}{a}\right)^{-\frac{2}{3}};$$

in the *Catenary*

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{c}, \text{ and } \frac{\rho'}{\rho} = \frac{c^2}{s^2};$$

and in the *Cycloid*

$$(\rho)^2 + (\rho')^2 = (4a)^2 = (\rho)^2 + (s)^2, \text{ and } \rho' = s.$$

19. It may be remarked that though it is perfectly immaterial where our origin is taken for finding the *Radial Curve*, the converse will not hold good, but when we wish to know the curve for which a given curve is the *Radial Curve*, the position of the origin must be assigned.

20. To find the curve of which the *Radial Curve* is $r\sin^3\theta = 2a$, we have, by (β),

$$2a\frac{d^2y}{dx^2} = 1,\ 2a\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{dy}{dx},\ a\left(\frac{dy}{dx}\right)^2 = y,$$

$$a^{\frac{1}{2}}y^{-\frac{1}{2}}\,dy = dx,\ 2a^{\frac{1}{2}}y^{\frac{1}{2}} = x;$$

therefore the curve required is $x^2 = 4ay$, a parabola with its axis vertical and latus rectum equal to $4a$.

21. If a parabola be the *Radial Curve*, its equation referred to the vertex being

$$r\sin^2\theta = 4a\cos\theta, \text{ or } r\sin^3\theta = 4a\cot\theta\sin^2\theta,$$

we have, by (β), for the determination of the corresponding curve,

$$\left(\frac{d^2y}{dx^2}\right)^{-1} = 4a\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-1},$$

$$\therefore \frac{2\frac{dy}{dx}\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2a},$$

$$\log\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \frac{x}{2a},\ \frac{dy}{dx} = \left(e^{\frac{x}{2a}} - 1\right)^{\frac{1}{2}}.$$

Assume $e^{\frac{x}{2a}} = 1 + z^2$; then

$$\frac{dy}{dz} = \frac{4az^2}{1+z^2},\ \therefore \frac{y}{4a} = z - \tan^{-1}z.$$

Hence the equation of the required curve is

$$\frac{y}{4a} = \left(e^{\frac{x}{2a}} - 1\right)^{\frac{1}{2}} - \tan^{-1}\left(e^{\frac{x}{2a}} - 1\right)^{\frac{1}{2}}.$$

22. Let the *Radial Curve* be the straight line $r\cos\theta = a$, then the corresponding curve is given by

$$\rho = a\sec\theta = a\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}},$$

$$\therefore \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{a}\frac{dy}{dx},$$

$$\therefore \tan^{-1}\left(\frac{dy}{dx}\right) = \frac{y}{a}, \text{ or } \frac{dy}{dx} = \tan\frac{y}{a};$$

hence the equation of the required curve is

$$\sin\frac{y}{a} = e^{\frac{x}{a}}.$$

23. If the *Radial Curve* be $r\tan\theta = a$, the corresponding curve is given by

$$\rho = a\cot\theta = a\frac{dy}{dx}, \text{ or}$$

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = a\frac{dy}{dx}\frac{d^2y}{dx^2},$$

$$\therefore a\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} = -x,$$

$$\therefore \frac{dy}{dx} = \frac{(a^2-x^2)^{\frac{1}{2}}}{x};$$

putting $x = a\sin\theta$, we have

$$\frac{dy}{d\theta} = a\frac{\cos^2\theta}{\sin\theta} = a(\operatorname{cosec}\theta - \sin\theta),$$

$$\therefore y = a(\log\tan\tfrac{1}{2}\theta + \cos\theta),$$

$$\text{or } y = (a^2-x^2)^{\frac{1}{2}} + a\log\frac{a-(a^2-x^2)^{\frac{1}{2}}}{x}.$$

The curve is, therefore, the *Tractory*.

1387 (Proposed by Mr. W. K. Clifford.)—Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be *opposite* intersections of the tangents, show that OA and OB are equally inclined to the tangent at O to the ellipse.

Solution by the PROPOSER.

We use *rectangular tangential* coordinates (Ferrers, Tril. Co., p. 130; Salmon, Higher Plane Curves, p. 2.) It is easily shown that the sum of the squares of the reciprocals of the intercepts made by any tangent to a circle on two diameters at right angles is constant. Hence the equation to a circle whose centre is the origin is

$$\xi^2+\eta^2 = c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The points $\xi = 0$, $\eta = 0$, are at an infinite distance, one on each of the axes; and $k = 0$ (where k is a constant) represents the origin. From this it follows that the equation

$$\xi^2 + bk\xi + ck\eta + dk^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots (2)$$

(where the k may be left out at pleasure) represents a conic touching the axis of ξ at the origin. For if we seek the tangents drawn from $k = 0$ to the curve, we find that they both coincide with the line $k\xi$, that is, with the axis of ξ. Now if we put

$$\xi^2+\eta^2-c^2 \equiv \mathrm{S},\ \xi + b\xi + c\eta + d \equiv \mathrm{T},$$

it is clear that the equation $\mathrm{S}+\lambda\mathrm{T} = 0$ represents an envelop of the second class, touching all the common tangents of S and T. The discriminant of this equation is of the third degree in λ; hence there are three values of λ for which $\mathrm{S}+\lambda\mathrm{T} = 0$ represents two points. But in every case the coefficient of $\xi\eta$ is zero; which is just the condition that the lines joining the origin to the two points (which are evidently opposite intersections of the common tangents) should be equally inclined to the axis of ξ. For if $a\xi + b\eta = 1$ be the equation of a point, (a, b) are its ordinary rectangular coordinates, and $(b : a)$ is the tangent of the angle which the line joining it to the origin makes with the axis of ξ. Hence if two points (a, b) and (c, d) are equally inclined to the point ξ, we must have

$$\frac{b}{a} = -\frac{d}{c}, \text{ or } ad+bc = 0;$$

but $(ad+bc)$ is the coefficient of $\xi\eta$ in the product $(a\xi+b\eta-1)(c\xi+d\eta-1)$. The theorem is therefore proved.

It will be observed that the discriminant being of the third degree in λ, must always have one real root; but there will be four real common tangents only when the conic is an ellipse cutting the circle in four points.

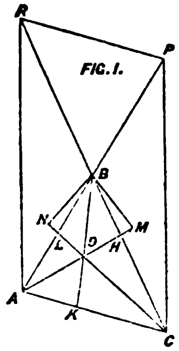
It appears therefore that *any* two conics have two *real* intersections of real or imaginary common tangents, corresponding to the centres of similitude of two circles.

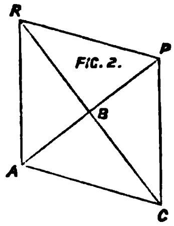
By projection we may show that "If a straight line A join the poles of B with respect to two conics, then the lines joining AB to a pair of opposite intersections of common tangents, form, with A, B, an harmonic pencil."

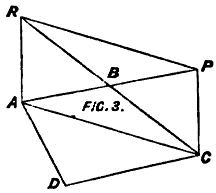
And by reciprocation,—"If a point A be the intersection of the polars of B with respect to two conics, and AB be cut by a pair of common chords in C, D, then ACBD is an harmonic range."

1402 (Proposed by Professor Sylvester, F.R.S., Royal Military Academy, Woolwich.)—Let A, B, C be three given points, O the centre of the circle passing through them, to find a point D, lying in the same circle with A, O, C, and such that its distances from A and C shall be in the duplicate ratio of the distances of B from the same.

Solutions (1—8) by Professor SYLVESTER; (9) by Mr. ARCHER STANLEY; (10) by Mr. J. R. WILSON, Jesus College, Cambridge; (11) by J. McDOWELL, B.A., F.R.A.S., Pembroke College, Cambridge; and (12) by Mr. W. K. CLIFFORD, and Mr. A. RENSHAW.







1. *Construction.*—Produce AB to P, and CB to R, making BP = BA, and BR = BC; join AR, CP, and make ∠BAD = BAR, and BCD = BCP; then D is the required point, such that DA : DC = BA^2 : BC^2, and D will lie on the circumference of the circle drawn through A, B, and O, the centre of the circle circumscribed about ABC.

2. *To show that* D *lies in the circle* AOC. Imagine BX, DZ drawn, towards the same parts, parallel to AR or CP; then ADZ = DAR = 2ABX, CDZ = DCP = 2CBX; hence, in fig. 1, ADC = 2ABC, and in fig. 3, ADC + 2ABC = 4 right angles: therefore in each case ADC = AOC, and D lies in the circle AOC.

3. The points O and D fall *inside* or *outside* the triangle ABC according as the angle ABC is *acute* (Fig. 1), or *obtuse* (Fig. 3). When ABC is a *right angle* (Fig. 2), O lies *in* AC, and the circumference on which D is to be found becomes the indefinite straight line AC. In this case D is evidently the foot of the perpendicular drawn from B to AC, but the construction for finding it fails, since the two lines AD, CD, which coincide, have no longer any definite intersection.

4. Proof by *Geometry* that
DA : DC = BA^2 : BC^2.

Let AD, BD, CD meet BC, CA, AB in H, K, L, and draw BM, BN perpendicular to AD, CD; then

AH : AR = HB : BR, CP : CL = PB : BL;

∴ AH : CL = AB . BH : CB . BL
= △ABH : △CBL;

therefore BM = BN, the angles HDL, ADC are *bisected* by BDK, and the triangles ABD, BDC are *similar;*

∴ DA : DC = AK : KC = △ABD : △BDC
= BA^2 : BC^2.

5. Proof by *Trigonometry.*

Let BAC = α, BCA = β, BAR = α', BCP = β'; then, obviously, $\alpha+\alpha'+\beta+\beta' = 180°$; also since

$$\frac{BC}{BA}=\frac{BR}{BA},\quad \frac{\sin\alpha}{\sin\beta}=\frac{\sin\alpha'}{\sin\beta'},\ \text{and}\ \frac{\sin\alpha'}{\sin\alpha}=\frac{\sin\beta'}{\sin\beta};$$

$$\therefore \frac{\sin(\alpha-\alpha')}{\sin(\beta-\beta')}=\frac{\sin(\alpha-\alpha')\sin(\alpha+\alpha')}{\sin(\beta-\beta')\sin(\beta+\beta')}$$

$$=\frac{\sin^2\alpha-\sin^2\alpha'}{\sin^2\beta-\sin^2\beta'}$$

$$=\frac{\sin^2\alpha}{\sin^2\beta}\cdot\frac{1-\dfrac{\sin^2\alpha'}{\sin^2\alpha}}{1-\dfrac{\sin^2\beta'}{\sin^2\beta}}=\frac{\sin^2\alpha}{\sin^2\beta};$$

∴ DA : DC = BA^2 : BC^2.

6. Proof by *Conic Sections.* Let a parabola be constructed passing through the points A, C, and touching the straight lines BA, BC. Then the axis of this parabola will be parallel to AR or CP, and thus each of the lines AD, CD will contain the focus, which will therefore be at D. Hence, by a *known property* of the parabola,

DA : DC = BA^2 : BC^2.

7. *Mechanical* Proof of the above assumed property of the parabola. Let AB represent in magnitude and direction the velocity of a projectile which describes the parabola touching AB, BC, at A and C. Then we may resolve this velocity into two represented by AC, CB. Also the force of gravity may be resolved in the directions of these two lines, and the motion parallel to either of them, say to CB, will be the same as if the other component velocity and component force did not act. Hence at C (at which point, by hypothesis, the motion takes place along BC) the velocity is the same in magnitude as the component at A parallel to this line, but reversed in direction. Hence BC represents the velocity at C in the same scale as AB at A.

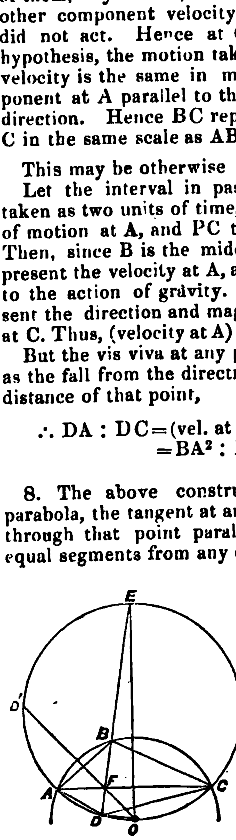
This may be otherwise proved as follows :—

Let the interval in passing from A to C be taken as two units of time, AP being the direction of motion at A, and PC the direction of gravity. Then, since B is the middle of AP, BP will represent the velocity at A, and PC the velocity due to the action of gravity. Hence BC will represent the direction and magnitude of the velocity at C. Thus, (velocity at A) : (vel. at C) = BA : BC.

But the vis viva at any point of the parabola is as the fall from the directrix, that is, as the focal distance of that point,

∴ DA : DC = (vel. at A)^2 : (vel. at B)^2
= BA^2 : BC^2.

8. The above construction shows that the parabola, the tangent at any point of it, and a line through that point parallel to the axis, cut off equal segments from any other tangent.



9. *Second Solution.*—Draw the diameter OE of the circle AOC, then the latter will be cut in the required point D by the line EB, produced if necessary.

For the angle ABC being the supplement of each of the equal angles ADB and BDC, it is evident that the

triangles ADB and BDC are equiangular; so that DA is to DC in the duplicate ratio of DA to DB, which is the same as the duplicate ratio of BA to BC.

A second point D′, possessing the required property, is found by producing the line which joins O and the point (F) where DE cuts AC; for (Euc. vi. 3) D′A : D′C = FA : FC = DA : DC $= BA^2 : BC^2$.

10. The angle ODB is a *right angle*, hence the point D may be otherwise determined by the intersection of AOC with the circle on OB as diameter.

11. *Third Solution.* Divide AC in F so that AF : FC in the duplicate ratio of BA : BC; join OF and produce it to meet the circle AOC in D′. D′ is the required point; for join D′A, D′C, then obviously OD′ bisects the angle AD′C,

$\therefore$ (Euc. vi. 3) D′A : D′C = AF : FC $= BA^2 : BC^2$.

12. Or, divide AC *internally* in F and *externally* in G, so that

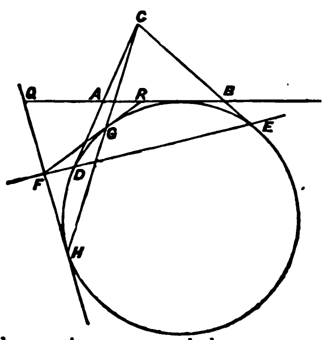
$$AF : FC = AG : GC = BA^2 : BC^2;$$

then the two required points D, D′ will obviously be determined by the intersections of AOC with the circle on FG as diameter.

1404 (Proposed by Mr. H. Murphy.)—A, B are two *fixed* points in a tangent to a given circle, and Q, R two *variable* points which form an *harmonic range* with A, B; required the locus of the intersection of tangents from Q, R to the circle.

Solution by Mr. W. K. Clifford; Mr. A. Renshaw; and Mr. H. Murphy.

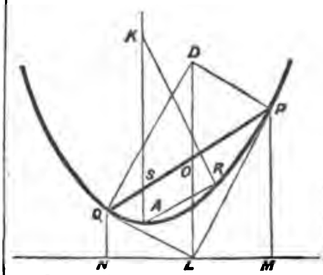
This is the reciprocal of Prop. 221 of McDowell's "*Exercises on Euclid and in Modern Geometry.*"



From A, B draw tangents CAD, CBE, and join DE; then DE will be the required locus. From any point F in DE, draw tangents FG, FH, then CGH is a straight line. Therefore HDGE is an harmonic range, and the tangents at those points cut any fifth harmonically; or Q, R, are *harmonic conjugates* to A, B.

1405 (Proposed by Mr. J. R. Wilson, Jesus College, Cambridge.)—A parabola is fixed with its vertex downwards and axis vertical, and within it is placed a smooth uniform rod. Give a geometrical construction for obtaining the position of the rod when in equilibrium.

Solutions (1) by Mr. W. K. Clifford; (2) by Mr. J. R. Wilson.



1. *First* Solution. With the focus (S) as centre, and radius equal to one-fourth the length of the rod, draw a circle cutting the parabola in two points (Y, Z, suppose) and the axis (SA) produced through A in X. Draw focal chords (PSQ, P′SQ′) parallel to XY, XZ, which are the *tangents* at Y, Z; then PSQ (or P′SQ′) will be the position of equilibrium required.

Let the *tangents* at P, Q intersect in L, and the *normals* in D. Then PSQ = 4SY = the *length* of the rod. And since L is a right angle, PLQD is a rectangle; therefore LD bisects PQ, and is *vertical*. But the force of gravity acts vertically through the middle (O) of PQ, and the resistance is along the normals; and as the lines of action of these forces pass through one point (D), the rod is in equilibrium. It will also rest perpendicular to the axis.

2. *Second* Solution. Let A be the vertex of the parabola PAQ, MN its directrix, PQ the rod in equilibrium, O its middle point. Draw PM, OL, QN perpendicular to the directrix. Then, since the rod is in equilibrium, its centre of gravity (O) must be in its *lowest position*; hence OL, or $\frac{1}{2}$ (PM + QN), that is $\frac{1}{2}$ (SP + SQ), is a *minimum*, which will be when the rod passes through the focus S.

Along the axis take AK equal to the length of the rod, and on AK as diameter describe a circle meeting the parabola in R; join AR, and through S draw PSQ parallel to AR, then PSQ will be the position of the rod.

For if AS $= a$, and RAS $= \theta$, we have

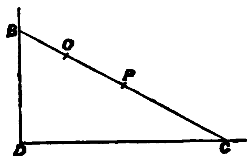
$$PQ = \frac{4a}{1-\cos^2\theta} = \frac{4a\cos\theta\sec\theta}{\sin^2\theta} = AR\sec\theta$$

$$= AK = \text{length of rod.}$$

1406 (Proposed by Mr. Alexander Renshaw.)—Referring to the figure for Question 1362, in the Educational Times for May; required the locus of the centre of the circle circumscribing the quadrilateral CDAB, the point D being a *fixed*,

and the point A a *variable* one in the rectangular hyperbola.

Solution by Mr. W. K. CLIFFORD; Mr. A. RENSHAW; and ALPHA.



The problem is equivalent to the following:—O, D are fixed points, DB and DC fixed straight lines, BC any straight line through O; required the locus of the *middle* (P) of the intercept (BOC).

Taking DC, DB as axes of x and y, let (h, k) be the given point O, and (x, y) the point (P) whose locus is required; then $DC=2x$, $DB=2y$, and

$$\frac{h}{2x}+\frac{k}{2y}=1;$$

hence the equation of the required locus is

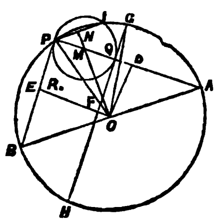
$$2xy=kx+hy.$$

The locus of P is therefore a rectangular hyperbola similar to the original, and similarly situated, having its centre at the middle of OD.

1350 (Proposed by W. S. B. Woolhouse, F.R.A.S., F.S.A., &c., London.)—If three points be taken at random in a given plane, the probability of their being the vertices of an acute triangle is $\frac{4}{\pi^2}-\frac{1}{8}$.

Solution by the PROPOSER.

The following solution is analogous to that given at page 234 of the January number, for a similar question in which the three points are taken in space. In the present case, the points being first supposed to be limited to the surface of a circle, radius unity, it may in like manner be shown that the value of the required probability will remain unaltered when one of the three points is fixed in the circumference.



In the diagram let P be this point; and Q a second and variable point. Produce PQ to meet the circumference again in A; on PQ as a diameter describe a circle (M) intersecting the given circle again in I. Then if PB, GH touch the circle (M) at P and Q, a triangle PQR will obviously be acute when the third point R is situated within that portion of the section of the given circle which lies between these two lines and which is not covered by the circle (M). Draw OM bisecting PI in N; demit the perpendiculars OD, OE on AP, BP; and join OP, OG. Let $PM=\rho$; the angle $POE=\alpha$; $GOE=\phi$; $PMN=\theta$; and $PON=\psi$.

Then the part of the circle (O) contained between PB, GH, being the difference between two segments HBG and BP, is

$$L=2\int d\phi\,\sin^2\phi-2\int d\alpha\,\sin^2\alpha;$$

the volume of the circle (M) is

$$M=\pi\rho^2;$$

and the portion of the circle (M) which lies without the given circle, being the difference between two segments which stand on PI, is

$$h-k=2\rho^2\int d\theta\,\sin^2\theta-2\int d\psi\,\sin^2\psi,$$

the initial values of the variables and integrals being severally zero.

The space to which the point R is limited, in the case of an acute triangle, is

$$V=L-M+h-k,$$

and the total space within which it can be taken is the given circle $(=\pi)$. Also the differential of the space which determines the number of points Q is $(2\rho)\,d\alpha\times d\,(2\rho)=4\rho d\rho d\alpha$; and it will be sufficient to take α from 0 to $\frac{1}{2}\pi$ so as to include one-half of the given circle.

The required probability may be thus expressed:

$$p=\frac{\iint V\,.\,4\rho d\rho d\alpha}{\iint \pi\,.\,4\rho d\rho d\alpha}=\frac{4\iint V\rho d\rho d\alpha}{\frac{1}{2}\pi^2}$$

$$=\frac{8}{\pi^2}\iint (L-M+h-k)\,\rho d\rho d\alpha$$

the respective terms of which we shall denote by (L), (M), (h), (k).

The limits of integration are

$\rho\ldots.0$ to $\cos\alpha$	$\psi\ldots.0$ to $\frac{1}{2}\pi-\alpha$
$\phi\ldots.\alpha$ to $\pi-\alpha$	$\alpha\ldots.0$ to $\frac{1}{2}\pi$;
$\theta\ldots.\alpha$ to $\frac{1}{2}\pi$	

and the integrals are obtained as follows:

$$\int L\rho d\rho=\tfrac{1}{2}\rho^2L-\tfrac{1}{2}\int\rho^2 dL$$

$$=\cos^2\alpha\,(\tfrac{1}{2}\pi-\alpha+\cos\alpha\,\sin\alpha)-\tfrac{1}{4}\int d\phi\,\sin^2\phi\,(\cos\alpha-\cos\phi)^2$$

$$=(\tfrac{3}{4}\cos^2\alpha-\tfrac{1}{16})\,(\tfrac{1}{2}\pi-\alpha+\cos\alpha\sin\alpha)+\tfrac{1}{8}\cos\alpha\,\sin^3\alpha;$$

$$\therefore\ (L)=\frac{8}{\pi^2}\int d\alpha\int L\rho d\rho=\frac{5}{16}+\frac{3}{\pi^2};$$

$$\int M\rho d\rho=\pi\int\rho^3 d\rho=\tfrac{1}{4}\pi\rho^4=\tfrac{1}{4}\pi\cos^4\alpha,$$

$$\therefore\ (M)=\frac{2}{\pi}\int d\alpha\cos^4\alpha=\frac{3}{8}:$$

$$\int h\rho d\rho=\int 2\rho^3 d\rho\int d\theta\,\sin^2\theta$$

$$=\tfrac{1}{2}\rho^4\int d\theta\,\sin^2\theta-\tfrac{1}{2}\int\rho^4 d\theta\,\sin^2\theta,$$

$$\therefore\iint h\rho d\rho d\alpha=\frac{3\pi^2}{128}-\frac{1}{2}\int d\alpha\int\rho^4 d\theta\,\sin^2\theta.$$

But $\int\rho^4 d\theta\,\sin^2\theta=\int d\theta\,\sin^2\theta\,(\cos\alpha-\sin\alpha\cot\theta)^4$

$=\frac{1}{2}\cos^4\alpha\,(\theta-\cos\theta\,\sin\theta)-2\cos^3\alpha\,\sin\alpha\,\sin^2\theta$

$+\,3\cos^2\alpha\,\sin^2\alpha\,(\theta+\cos\theta\,\sin\theta)$

$-\,4\cos\alpha\,\sin^3\alpha(\frac{1}{2}\cos^2\theta+\log\sin\theta)$

$$+\sin^4\alpha\left(\frac{\cos^3\theta}{2\sin\theta}-\frac{3\cos\theta}{2\sin\theta}-\frac{3}{2}\theta\right)$$

$=(\frac{1}{2}\pi-\alpha)\,(\frac{1}{2}\cos^4\alpha+4\cos^2\alpha\sin^2\alpha-\frac{3}{2}\sin^4\alpha)-\frac{3}{2}\cos^3\alpha\sin\alpha+\frac{3}{2}\cos\alpha\sin^3\alpha+4\cos\alpha\sin^3\alpha\log\sin\alpha;$

$$\therefore \int d\alpha\int\rho^4 d\theta\sin^2\theta=\tfrac{1}{4},$$

$$\therefore \iint h\rho d\rho d\alpha=\frac{3\pi^2}{128}-\frac{1}{8},$$

$$\text{and } (h)=\frac{8}{\pi^2}\left(\frac{3\pi^2}{128}-\frac{1}{8}\right)=\frac{3}{16}-\frac{1}{\pi^2}.$$

Lastly, $\int k\rho d\rho=\frac{1}{2}\rho^2 k-\frac{1}{2}\int\rho^2 dk=$

$\frac{1}{2}\cos^2\alpha\,(\frac{1}{2}\pi-\alpha-\cos\alpha\sin\alpha)-\int\rho^2 d\psi\sin^2\psi;$

$$\therefore \iint k\rho d\rho d\alpha=\frac{\pi^2}{32}-\int d\alpha\int\rho^2 d\psi\sin^2\psi.$$

But $d\psi=d\theta$, $\sin\psi=\rho\sin\theta$;

$$\therefore \int d\alpha\int\rho^2 d\psi\sin^2\psi=\int d\alpha\int\rho^4 d\theta\sin^2\theta$$

$$=(\text{as before})\ \tfrac{1}{4};$$

$$\iint k\rho d\rho d\alpha=\frac{\pi^2}{32}-\frac{1}{4};$$

$$\text{and } (k)=\frac{8}{\pi^2}\left(\frac{\pi^2}{32}-\frac{1}{4}\right)=\frac{1}{4}-\frac{2}{\pi^2}.$$

Hence,

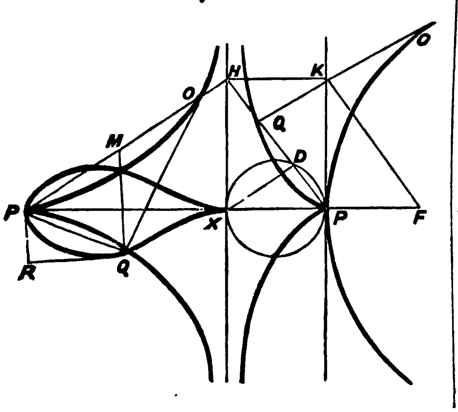
$$p=(\mathrm{L})-(\mathrm{M})+(h)-(k)$$

$$=\left(\frac{5}{16}+\frac{3}{\pi^2}\right)-\frac{3}{8}+\left(\frac{3}{16}-\frac{1}{\pi^2}\right)-\left(\frac{1}{4}-\frac{2}{\pi^2}\right)$$

$$=\frac{4}{\pi^2}-\frac{1}{8}.$$

1390 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—Find the locus of the foot of a perpendicular drawn from the vertex on a tangent to the Cissoid; trace the curve, and find its length and area.

Solution by the Proposer.



1. It will be convenient to have a distinctive name for the curve which is the locus of the foot of a perpendicular drawn from a fixed point on a variable tangent to a given curve; I shall therefore use, for this purpose, the term *Pedal Curve*, which was proposed by Dr. Salmon ("Geometry of Three Dimensions," Art. 440), and has been adopted by Dr. Hirst, in a paper ("On the volumes of Pedal Surfaces") recently published in the Transactions of the Royal Society.

The fixed point may be appropriately called the *pedal origin*; and a point on the *Pedal* may be said to *correspond* to the *point of contact* of the tangent drawn from it to the given curve.

Again, the curve whose *Pedal* is the given curve, or, what is the same thing, the *envelop* of a perpendicular at the end of a radius vector to the curve, may be called a *negative Pedal*; and by a process similar to that by which this and the former *Pedal* have been derived from the given curve, we may obtain, from these, *two series* of Pedals, which may be distinguished as *first*, *second*, &c., *positive* or *negative Pedals*.

2. In a paper in the *Educational Times* for July, 1859, I proved the following theorem, from which it will be seen that *positive*, as well as negative *Pedals* may also be regarded as *envelops*.

"From a fixed point (P) let perpendiculars (PQ) be drawn to the tangents (OQ) to a given curve (C), and call (L_1) the locus of Q; also let (L_2) be the locus of the ultimate intersections (or envelop) of a series of consecutive circles which pass through P, and have their centres (O) in the given curve (C); and let (L_3) be a like locus in reference to circles on radii (PO) as diameters: then will (L_1), (L_3) be one and the same curve, and (L_1), (L_2) will be similar and similarly situated, every radius of (L_2) being double the corresponding radius of (L_1)."

The curve (L_1) is what is here called the *first positive Pedal* of the given curve (C).

From this it is evident that the *normal* (QM) at Q to the Pedal, being also the normal to the circle (OPQ) which the Pedal touches at Q, must pass through the *centre* of this circle, that is, through the *middle* (M) of the radius PO.

Hence, if O, Q, R be *corresponding points* on a curve and its *first* and *second* positive Pedals, QR will be a tangent at Q to the first Pedal, and the perpendicular (PR) from P on this tangent must be parallel to MQ;

$$\therefore \angle\,\mathrm{OPR}=\mathrm{OMQ}=2\mathrm{OPQ};$$

and so on for Pedals of higher orders, if Z be the point on the nth Pedal corresponding to O on the given curve, we shall have

$$\angle\,(\mathrm{OPZ})=n\,(\mathrm{OPQ}).$$

The right-angled triangles OPQ, QPR, &c., are therefore similar, and the radii (PO, PQ, PR...... PZ) from the pedal origin to *corresponding points* (O, Q, R, ... Z) in the series of curves are continual proportionals.

3. Referred to rectangular axes through the pedal origin, let (α, β) be the coordinates of a point (O) on the given curve, and (x, y) those of the corresponding point (Q) on the *first positive Pedal*; then, putting λ for ($d\beta : d\alpha$), the *locus* of (x, y), that is, the Pedal, may be found by eliminating α, β, from the three equations,

$$f(\alpha, \beta)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$(y-\beta)=\lambda (x-\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$x+\lambda y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

(1) being the equation of the given curve, (2) that of the tangent (OQ) to it at the point (α, β), and (3) the perpendicular (PQ) from the origin on the tangent.

From (1), (2), (3) we have

$$x=\frac{\lambda (\lambda\alpha-\beta)}{1+\lambda^2},\ y=\frac{\beta-\lambda\alpha}{1+\lambda^2} \quad \ldots\ldots\ldots\ldots (4)$$

which will immediately give the Pedal, by expressing the coordinates (x, y) of any point on it as functions of a third variable, if (α, β) can be expressed in terms of this variable.

4 Again, let (ϱ, ϕ) and (r, θ) be corresponding *polar coordinates* of O and Q, and s the length of an arc of the Pedal, estimated from some fixed point on it; then $\angle$ POQ=PQR, (Art. 2),

$$\therefore \frac{r}{\varrho} = \sin \text{POQ} = \sin \text{PQR} = \frac{rd\theta}{ds};$$

hence the *length* of the Pedal may be found from the expression

$$s=\int \varrho d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

5. Putting $\text{PR}=p$, the similar triangles OPQ, QPR give

$$p\varrho=r^2, \text{ or } p=\frac{r^2}{\varrho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now (ϱ, r) and (r, p) are the radius vector and perpendicular from the origin on the tangent at corresponding points (O, Q) of a curve and its first positive Pedal; hence (6) is the same as the formula deduced by Todhunter in Art. 329 of his *Differential Calculus*, and again in Art. 91 of his *Integral Calculus*, where he calls it "a theorem of some interest in the Differential Calculus." If a curve be defined by an equation between the radius vector (r) of any point in it, and the perpendicular (p) from the origin on the tangent at that point, we may, by (6), obtain similar equations for *the whole series* of positive and negative *Pedals*.

Thus, if $f(r, p)=0$ be the equation of a curve, the equation of the *first positive Pedal* will be

$$f\left(\frac{r^2}{p}, r\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and that of the *first negative Pedal*

$$f\left(p, \frac{p^2}{r}\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The process may be easily extended to Pedals of higher orders.

6. The focus being the origin, the equation in (r, p) of the ellipse is

$$b^2r=(2a-r)\,p^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9);$$

hence, by (7), the equation of the *first positive Pedal* will be

$$r^2+b^2=2ap \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

which is easily seen to be that of a circle on the major axis as diameter.

Also, by (8), the equation of the *first negative Pedal* will be

$$b^2r^2=(2a-p)\,p^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

which agrees with the result otherwise obtained in equation (11) of my Solution of Quest. 1283, in the *Educational Times* for December 1862, where the form and properties of the curve (11), its rectification and quadrature, are fully investigated.

7. The *second positive focal Pedal* of an ellipse is a curve which may be generated in the following manner. Construct a circle on the distance of the pedal origin from the centre as diameter; and on every radius vector, drawn from the origin to a point in the circumference, take both ways from this point a line equal to the major semi-axis; then the locus of the ends of the line thus obtained will obviously be the *first* Pedal of the circle circumscribing the ellipse, and therefore the *second* Pedal of the ellipse itself.

The construction also gives the Pedal of a circle, the origin being any point in its plane, and shows that when the origin is on the circumference, the *Pedal* is a *Cardioid*.

The first *negative*, and the first and second *positive*, focal Pedals of an hyperbola, may be obtained and constructed in like manner to those of the ellipse.

8. When the pedal origin is the vertex of an ellipse, it is shown in Art. 8 of my paper before referred to, that the equation of the Pedal is

$$(x^2+y^2-ax)^2=a^2x^2+b^2y^2 \quad \ldots\ldots\ldots\ldots (12),$$

$$\text{or, } r=a\cos\theta \pm (a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}} \quad ..(13).$$

Putting h for the principal semi-parameter, (12) may be written

$$\frac{(x^2+y^2)^2}{a}=2x\,(x^2+y^2)+hy^2 \quad \ldots\ldots\ldots (14).$$

When $a=b$, (13) becomes

$$r=a\,(\cos\theta \pm 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

the equation of a *Cardioid* (see Art. 7).

Supposing a to increase without limit, we shall obtain from (14) the corresponding Pedal of the *Parabola*; hence, writing now $2a$ for h, so that a is the distance of the vertex from the focus, and putting $(-\alpha, \beta)$ for (x, y), the equation becomes

$$(a-\alpha)\,\beta^2 = \alpha^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16),$$

which shows that the Pedal is a *Cissoid*.

The corresponding polar coordinates being (ϱ, ϕ), the *Cissoid* may be defined by either of the following systems of equations:—

$$\varrho=a\sin\phi\tan\phi=a\,(\sec\phi-\cos\phi) \quad ..(17)$$

$$\alpha=a\sin^2\phi,\ \beta=a\sin^2\phi\tan\phi \quad \ldots\ldots (18).$$

9. The Pedal deduced in the last Art. may be simply obtained by *geometry*.

For let P (right-hand figure) be the vertex and F the focus of a parabola, PK the tangent at the vertex, OKQ any other tangent, and PDQH a perpendicular from P on OKQ, meeting the directrix (XH) in H, and the circle on XP in D.

Then PK is the *focal Pedal*, as is well known; therefore FKQ is a right angle, and FK is parallel to PH; hence it is easy to see that the triangles FPK, PXH are equal in all respects, and likewise QHK, DPX; therefore HQ=PD, which shows that the *locus* of Q, that is, the Pedal for the origin P, is a *Cissoid* having PDX for its generating circle, and XH for its *asymptote*.

10. The Problem in the Question is, therefore,

to find the *form, length,* and *area,* of the *second positive Pedal* of a Parabola, the vertex being the pedal origin.

Taking the equations (18) of the Cissoid, and referring to the left-hand figure, where PO $= \rho$, $\angle$ OPX$=\phi$, we find, by (4), (6), (17),

$$\lambda = \frac{\sin\phi\,(2\cos^2\phi+1)}{2\cos^3\phi} \quad\ldots\ldots\ldots\ldots\ldots(19)$$

$$x = \frac{a\sin^4\phi\,(2\cos^2\phi+1)}{3\cos^2\phi+1} \quad\ldots\ldots\ldots\ldots(20)$$

$$y = \frac{-2a\sin^3\phi\cos^3\phi}{3\cos^2\phi+1} \quad\ldots\ldots\ldots\ldots\ldots(21)$$

$$r^2 = \frac{a^2\sin^6\phi}{3\cos^2\phi+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$\tan\theta = \frac{-2\cos^3\phi}{\sin\phi\,(2\cos^2\phi+1)} \quad\ldots\ldots\ldots\ldots\ldots(23)$$

$$\frac{d\theta}{d\phi} = \frac{6\cos^2\phi}{3\cos^2\phi+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

$$p = \frac{a\sin^4\phi\cos\phi}{3\cos^2\phi+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

$$\frac{dp}{d\phi} = \frac{a\sin^3\phi\,(\cos^2\phi+1)\,(9\cos^2\phi-1)}{(3\cos^2\phi+1)^2} \quad\ldots(26).$$

11. From (23), $\tan^3\phi + 3\tan\phi + 2\cot\theta = 0$, whence $\tan\phi = (\tan\tfrac{1}{2}\theta)^{\frac{1}{3}} - (\cot\tfrac{1}{2}\theta)^{\frac{1}{3}} \ldots\ldots\ldots(27).$

Again, from (23) and (22), we have

$$\sec^2\theta = \frac{3\cos^2\phi+1}{\sin^2\phi\,(2\cos^2\phi+1)^2} = \frac{a^2\sin^4\phi}{r^2(2\cos^2\phi+1)^2},$$

$$\text{whence } \frac{a}{r}\cos\theta = 3\cot^2\phi + 1 \quad\ldots\ldots\ldots\ldots(28).$$

$$(23)\times(28) \text{ gives } \frac{a}{r}\sin\theta = -2\cot^3\phi \quad\ldots\ldots(29).$$

Eliminating ϕ from (27), (29), and also from (28), (29), we have

$$r = \left\{(\cos\tfrac{1}{2}\theta)^{\frac{2}{3}} - (\sin\tfrac{1}{2}\theta)^{\frac{2}{3}}\right\}^3 \quad\ldots\ldots\ldots\ldots(30)$$

$$\text{and } 4\,(a\cos\theta - r)^3 = 27a^2r\sin^2\theta \quad\ldots\ldots\ldots(31)$$

$$\text{or, } 4\,(x^2+y^2-ax)^3 + 27a^2y^2\,(x^2+y^2) = 0 \quad\ldots(32)$$

12. Of these equations, (20) – (24) express (x, y) and (r, θ) as functions of ϕ, (30) or (31) is the equation of the curve in *polar coordinates* (r, θ), and (32) the corresponding equation in *rectangular coordinates* (x, y).

From (26) we see there is a *point of inflexion* when $\cos\phi = \frac{1}{3}$, or $a = \frac{8}{9}a$, $x = \frac{176}{243}a$; and from (21) the *ordinate* (y) is found to be a *maximum* when $\sin\phi = \frac{1}{4}(\sqrt{17}-1)$. The Pedal is of the form shown in the diagram, and has a *ceratoid cusp* at X.

13. Putting L for the *whole length* of the curve, we have, by (5), (17), (24),

$$\frac{s}{a} = \int \frac{6\sin^2\phi\cos\phi\,d\phi}{3\cos^2\phi+1}$$

$$= \frac{2}{\sqrt{3}}\log\left(\frac{2+\sqrt{3}\sin\phi}{2-\sqrt{3}\sin\phi}\right) - 2\sin\phi;$$

$$\therefore\ \mathrm{L} = (2s, \text{ from } \phi = 0 \text{ to } \phi = \tfrac{1}{2}\pi)$$

$$= \frac{8a}{\sqrt{3}}\log(2+\sqrt{3}) - 4a.$$

14. If A be the *whole area* of the curve, we have, by (22), (24),

$$\mathrm{A} = \int_0^{\frac{1}{2}\pi} \frac{6a^2\sin^6\phi\cos^2\phi\,d\phi}{(3\cos^2\phi+1)^2}$$

$$\therefore\ \frac{\mathrm{A}}{a^2} = \int_0^{\frac{1}{2}\pi} \sin^5\phi\cos\phi\,d\,(3\cos^2\phi+1)^{-1}$$

$$= \left\{\frac{\sin^5\phi\cos\phi}{3\cos^2\phi+1} + \int \frac{(6\sin^6\phi - 5\sin^4\phi)\,d\phi}{4-3\sin^2\phi}\right\}_{\phi=0}^{\phi=\frac{1}{2}\pi}$$

$$= \int_0^{\frac{1}{2}\pi} d\phi\,\left\{\frac{\frac{16}{3}}{3\cos^2\phi+1} - 2\sin^4\phi - \sin^2\phi - \tfrac{4}{3}\right\}$$

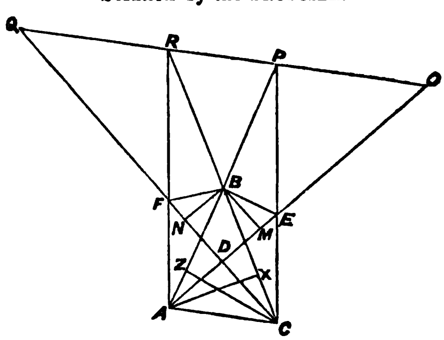
$$\text{and } \tfrac{16}{3}\int\frac{d\phi}{3\cos^2\phi+1} = \tfrac{8}{3}\tan^{-1}(\tfrac{1}{2}\tan\phi);$$

$$\therefore\ \frac{\mathrm{A}}{a^2} = \tfrac{4}{3}\pi - \tfrac{3}{8}\pi - \tfrac{1}{4}\pi - \tfrac{2}{3}\pi,$$

$$\therefore\ \mathrm{A} = \tfrac{1}{24}\pi a^2.$$

1402 (Proposed by Professor Sylvester, F.R.S., Royal Military Academy, Woolwich.)—Let A, B, C be three given points, O the centre of the circle passing through them, to find a point D, lying in the same circle with A, O, C, and such that its distance from A and C shall be in the duplicate ratio of the distances of B from the same.

Solution by the Proposer.



Construction.—Produce AB to P, and CB to R, making BP = BA, and BR = BC; join AR, CP, and make $\angle$ BAD=BAR, and BCD=BCP; then D will be the point required.

It is shown on p. 20 that D lies in the

circle through A, O, C ; and we now give a *second Proof by Geometry*, that

$$DA : DC = BA^2 : BC^2.$$

Demonstration.—Draw AX, CZ perpendicular respectively to BC, BA, and produce the lines as in the diagram.

$$\text{Then } AO : OP = AR : RP = CP : PR$$
$$= CQ : RQ,$$
$$\therefore OP : RQ = AO : CQ = DA : DC.$$

Again, it is easily seen that AX, CZ are parallel respectively to BF, BE ;

$$\therefore OP : AC = PE : EC = PB : BZ,$$
$$\text{and } AC : RQ = AF : FR = BX : BR ;$$
$$\therefore OP : RQ = BA . BX : BC . BZ = BA^2 : BC^2,$$
$$\text{or, } DA : DC = BA^2 : BC^2.$$

Note.—By the mechanical method of proof used in Art. 7 of the solution of this Question given on p. 20, it may be easily shown that the perpendiculars (BM, BN) from B on DA, DC are equal to each other. For RA, PC being the direction of gravity, the perpendiculars on RA, PC from B, which are respectively equal to BM, BN, represent the *horizontal velocities* at A, C, on the same scale as AB, BC represent the *total velocities* at these points; therefore BM = BN. Hence, as shown in the solution referred to (Art. 4), BD will *bisect* the angle ADC, whence we obtain a proof of the known theorem, that "the line joining the focus of a parabola to the intersection of two tangents bisects the angle subtended at the focus by their chord of contact."

1331 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—When $e = 1$, find the value of

$$\frac{(13-26e^2+16e^4)(4e^2-1)^{\frac{1}{2}}}{(1-e^2)^2} - \frac{9e^2-6}{(1-e^2)^{\frac{5}{2}}}\cos^{-1}\frac{2e^2-1}{e}.$$

Solution by the Proposer.

1. By the formula

$$\sin^{-1} s = s + \frac{1}{2}\frac{s^3}{3} + \frac{1 . 3}{2 . 4}\frac{s^5}{5} + \dots.$$

we have, in the given expression,

$$\frac{1}{(1-e^2)^{\frac{5}{2}}}\cos^{-1}\frac{2e^2-1}{e} =$$

$$\frac{1}{(1-e^2)^{\frac{5}{2}}}\sin^{-1}\frac{(1-e^2)^{\frac{1}{2}}(4e^2-1)^{\frac{1}{2}}}{e} =$$

$$\frac{(4e^2-1)^{\frac{1}{2}}}{e(1-e^2)^2} + \frac{(4e^2-1)^{\frac{3}{2}}}{6e^3(1-e^2)} + \frac{3(4e^2-1)^{\frac{5}{2}}}{40e^5} + F(e),$$

where $F(e)$ is a series of terms, each of which contains $(1-e^2)$ as a factor.

Hence the expression becomes, successively,

$$\frac{(4e^2-1)^{\frac{1}{2}}}{40e^5(1-e^2)^2}\Big\{40e^5(13-26e^2+16e^4) - 120e^4(3e^2-2)$$
$$- 20e^2(3e^2-2)(1-e^2)(4e^2-1)$$
$$- 9(3e^2-2)(1-e^2)^2(4e^2-1)^2\Big\} + F(e) ;$$

$$\frac{(4e^2-1)^{\frac{1}{2}}}{40e^5(1-e^2)^2}\Big\{18-247e^2+1364e^4+520e^5-2431e^6$$
$$-1040e^7+1608e^8+640e^9-432e^{10}\Big\} + F(e) ;$$

$$\frac{(4e^2-1)^{\frac{1}{2}}}{40e^5(1+e)^2}\Big\{18+36e-193e^2-422e^3+713e^4$$
$$+2368e^5+1592e^6-224e^7-432e^8\Big\} + F(e) ;$$

the last fraction being obtained by dividing both terms of the preceding by $(1-e)^2$.

Putting now $e = 1$, the last fraction gives

$$\frac{108}{5}\sqrt{3} = \text{the value required.}$$

2. The expression in the Question occurs in finding the area of the *first negative focal Pedal* of an ellipse. For it is shown in my Solution of Question 1283, in the *Educational Times* for Dec. 1862, that if h be the principal semi-parameter of an ellipse, and its eccentricity (e) be greater than $\frac{1}{2}$, the *negative Pedal* will consist of a *curvilinear triangle*, together with a *loop* whose area is $\frac{1}{8}h^2L$, where L is the expression in the Question.

If we suppose the parameter ($2h$) to remain *constant*, and the eccentricity (e) to increase up to 1, the *ellipse* will become a *parabola*, the *curvilinear triangle* will be transformed into two *infinite branches*, and the area of the *loop* will be

$$\left(\frac{18}{5}\sqrt{3}\right)h^2.$$

3. In Art. 7 of the Solution of Question 1283, it is shown that, when $e > \frac{1}{2}$, the *length* of the loop of the *negative Pedal* of the ellipse is $2hL'$, where

$$L' = 2(4e^2-1)^{\frac{1}{2}} + \frac{1}{(1-e^2)^{\frac{1}{2}}}\cos^{-1}\frac{2e^2-1}{e}.$$

By the method used above, we find

$$L' = \left(2+\frac{1}{e}\right)(4e^2-1)^{\frac{1}{2}} + F'(e);$$

where $F'(e)$ involves $(1-e^2)$ as a factor, and vanishes when $e = 1$. Hence the length of the loop of the *negative Pedal* of a parabola is

$$(6\sqrt{3})\,h.$$

4. Referred to the focus as origin, and the line (a) joining the focus to the vertex as *positive* axis of x, the *rectangular* equation of the *negative focal Pedal* of a parabola is (a being $= \frac{1}{4}h$)

$$27a(x^2+y^2) = (4a-x)^3,$$

and the corresponding *polar* equation is

$$r = a\sec^3\tfrac{1}{3}\theta.$$

From the polar equation we should find that the

length and *area* of the *loop* of the Pedal are

$$2a\int_0^\pi \sec^4 \tfrac{1}{2}\theta\, d\theta, \quad a^2\int_0^\pi \sec^6 \tfrac{1}{2}\theta\, d\theta,$$

$$\text{or } (12\sqrt{3})\,a, \quad \left(\frac{72}{5}\sqrt{3}\right) a^2;$$

which agree with the results obtained above.

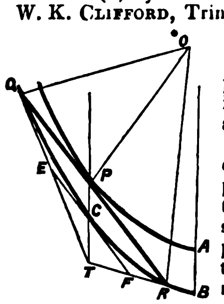
5. If this curve (the *focal negative Pedal* of a parabola) be supposed to consist of matter which attracts according to the inverse square of the distance, and A_1, A_2, A_3 be the attractions, on a particle at the focus, of the *whole* curve, the *loop*, and the part cut off by the parameter of the parabola (that is, the arc which exerts the *greatest* resultant attraction towards the vertex), it is shown in my Solution of Question 1341, in the *Educational Times* for March 1863, that

$$A_1 : A_2 : A_3 = 8 : 6\sqrt{3} : 19.$$

Moreover, an arc of the curve, extending to an angular distance of 23° 44′ 2″ on each side of the axis, would exert the same attraction as the whole curve on a particle at the focus of the parabola.

1393 (Proposed by Mr. J. R. Wilson, Jesus College, Cambridge.)—A shell formed of two equal paraboloids of revolution, having a common axis, is fixed with its vertex downwards, and axis vertical; and a heavy uniform rod of given length rests within it, in a vertical plane through the axis. Compare the pressures on the lower surface of the shell.

Solutions (1) by Mr. J. R. Wilson; (2) by Mr. W. K. Clifford, Trinity College, Cambridge.



1. *First Solution.* — Suppose the plane of the paper to be that containing the rod and common axis of the paraboloids.

Let QR be the rod in equilibrium, and P the point in which it touches the parabola AP; then, since QR is manifestly parallel to the tangent at the extremity of the diameter through P of the parabola BR, it is an ordinate of this diameter and therefore P is the middle point of the rod.

Now the rod is acted on by the normal force, F, F′ at Q, R, the weight of the rod acting vertically through P, and the reaction of the parabola AP at P. Thus, drawing the tangents QT, RT, and taking moments about P, we have

$$F : F' = \cos TRP : \cos TQP.$$

If $QR = 4c$, $AB = k$, $4a$ = latus rectum of a parabola, θ = inclination of QR the axis, it may be easily shown that $TP = 2k$, $c^2 \sin^2\theta = ak$, and

$$\frac{\cos TRP}{\cos TQP} = \frac{TQ\,(c - k\cos\theta)}{TR\,(c + k\cos\theta)}.$$

It will thus be seen that

$$\frac{F}{F'} = \left\{\frac{c^2 - k\,(c^2 - ak)^{\frac{1}{2}}}{c^2 + k\,(c^2 - ak)^{\frac{1}{2}}}\right\}\left\{\frac{c^2 + k^2 + 2k\,(c^2 - ak)^{\frac{1}{2}}}{c^2 + k^2 - 2k\,(c^2 - ak)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

2. *Second Solution.*—Let QPR be the rod, of length. $4c$; draw tangents QT, RT, and normals QO, RO, to the outer parabola. We know by geometry that QP = PR, and therefore PT is vertical. Now the rod is kept at rest by four forces, two of which, viz., gravity and the resistance at P, pass through P; therefore the resultant of the pressures at Q, R acts along OP. But OP, bisecting QR, is half the diagonal of the completed parallelogram (OQ, OR); hence the resistances at Q, R are as the normals OQ, OR; that is, as sin ORQ : sin OQR, or as cos TRP : cos TQP. Now draw a tangent (ECF) to the outer parabola at the point (C) where TP meets it; then, putting $AB = k$, $4a$ = principal parameter of BRQ, and $\angle$ ECP = θ, the equation of BRQ, referred to CP, CQ will be $y^2 = \dfrac{4a}{\sin^2\theta}\,x$......(1).

At the point P, $x = k$, $y = 2c$; $\therefore c^2 \sin^2\theta = ak$..(2).

The tangents QT, RT are represented by

$$k^2 y^2 = c^2\,(x + a)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

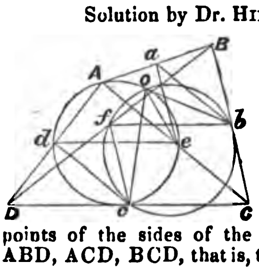
Therefore by the usual formula for oblique axes

$$\frac{\cos TFC}{\cos TEC} = \left(\frac{c - k\cos\theta}{c + k\cos\theta}\right)\left(\frac{c^2 + k^2 + 2ck\cos\theta}{c^2 + k^2 - 2ck\cos\theta}\right)^{\frac{1}{2}}.$$

This, therefore, is the ratio required.

1408 (Proposed by Matthew Collins, B.A., Senior Moderator in Mathematics and Physics, Trinity College, Dublin.)—Five points P.ABCD being taken at random on a plane, through P and any two of the other four points ABCD, six circles can be described, which will form four circular triangles ABC, ABD, ACD, BCD, of which P is none of the corners; on each side of each of these four circular triangles take a point which is the harmonic conjugate of P, relative to the extremities of this side (thus P′, lying between A and B, is the harmonic conjugate of P relative to A and B); then prove that the four circles passing through the three points P′ (harmonic conjugates of P) lying on the sides of each of the four circular triangles above mentioned will intersect in one point.

Solution by Dr. Hirst, F.R.S.



1. The *inverse* of this theorem may be thus enunciated and proved.

If A, B, C, D be any four points in a plane, the four circles drawn, respectively, through the middle points of the sides of the four triangles ABC, ABD, ACD, BCD, that is, the *Nine Point Circles*

of these triangles, will all pass through one and the same point.

Let a, b, c, d, e, f, be the middle points of the sides and diagonals of the quadrilateral ABCD, and let two of the circles in question, (cde) and (cbf), meet in o; then, since fb and de are parallel, the angle boe is equal to the difference of the angles deo and fbo, which latter are respectively equal to dco and fco, whose difference dcf is manifestly equal to bae.

The circle (bae) therefore passes through o. In a similar manner the fourth circle (dfa) may be proved to pass through o.

2. If the *inverse* of the whole figure be taken with respect to any point P in its plane, each side, such as AB, will correspond to the arc of a circle passing through P, and its middle point (a) to the harmonic conjugate, on that arc, of P, with respect to A and B.

The four circles (abe), (afd), (cde), (cfb) will correspond to other circles which, like the original, will all pass through one and the same point, the *inverse* of o.

The original theorem, therefore, is established.

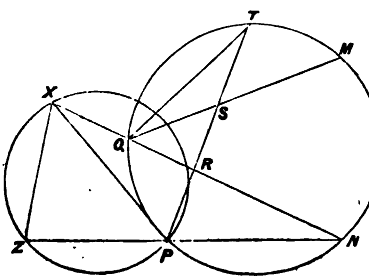
[NOTE.—We have received from Dr. Salmon the following Question, which includes the theorem proved in Art. 1 of the preceding Solution:—

"From four points can be made four triangles: their *Nine Point Circles* pass through a common point, viz., the centre of the equilateral hyperbola determined by the four points."

Dr. Salmon's theorem may be proved by showing, as is easily done, that the *locus* of the centre of an *equilateral hyperbola* described about a *triangle* is the *Nine Point Circle* of the triangle; and thus the centre of an equilateral hyperbola described about a *quadrilateral* must be at the point of intersection of any two of the *Nine Point Circles* of the four triangles into which the quadrilateral may be divided.—ED.]

1412 (Proposed by T. T. Wilkinson, F.R.A.S.)—Let P, Q, be given points in the circumference of a circle given in magnitude and position; QM, QN, right lines given in position. It is required to draw from P a line PRST, meeting QN in R, QM in S, and the circle in T, so that RS may have to ST a given ratio.

Solutions (1) by the EDITOR; (2) by Mr. W. HOPPS; Mr. H. MURPHY; Mr. J. WILSON; and the PROPOSER.



1. *First Solution.*—Produce NP to Z, so that

$$ZP : PN = RS : ST = \text{the given ratio};$$

on PZ construct a circular segment (ZXP) *similar* to MQN, cutting NQ in X; join ZX, and draw PRST, making $\angle$ TPN = ZXN; then PRST will be the line required.

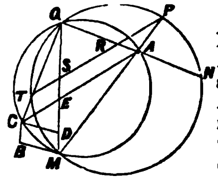
For join XP, QT;

then $\angle$ TQR = TPN = ZXN;

also $\angle$ QTR = XNZ, and RQS = ZXP;

hence the triangles TQR, RQS, SQT are *similar* to NXZ, ZXP, PXN;

$\therefore$ RS : ST = ZP : PN = the given ratio.



2. *Second Solution.*—Draw PM cutting QN in A; also draw MB parallel to AQ, making AQ : MB = the given ratio. Draw BC parallel to QM, meeting the circle through Q, A, M, in C, and join QC, cutting the given circle in T. Lastly, draw TP cutting QM, QN in S, R, and the problem is solved. For, join CA, cutting QM in E, and through C draw a line parallel to AQ, or MB, meeting QM in D. Then $\angle$ QTP = QMA = QCA, therefore CA is parallel to TP; hence by similar triangles and parallels RS : ST = AE : EC = AQ : CD = AQ : MB = the given ratio by construction.

1418 (Proposed by M. A. Vectensis.)—The angular points and sides of an acute-angled triangle are taken as the centres and directices of three ellipses, which have a common focus coinciding with the point of intersection of the perpendiculars from the angles on the sides; if (a_1, a_2, a_3), (b_1, b_2, b_3), (e_1, e_2, e_3), (h_1, h_2, h_3) be the respective semi-major and semi-minor axes, the eccentricities, and semi-parameters of the ellipses, and R the radius of the circumscribing circle of the triangle, prove that

(1) $a_1^2 + a_2^2 + a_3^2 = \frac{1}{2}(a^2 + b^2 + c^2)$.

(2) $b_1^2 = b_2^2 = b_3^2 = 4R^2 \cos A \cos B \cos C$.

(3) $h_1^2 + h_2^2 + h_3^2 = 4R^2 \cos A \cos B \cos C$.

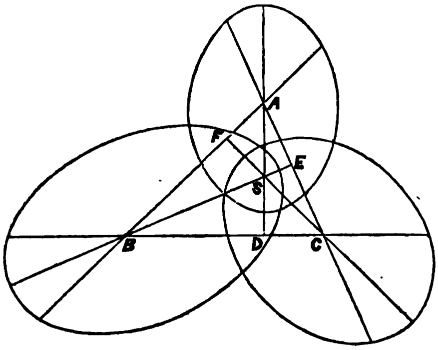
(4) $e_1^2 + e_2^2 + e_3^2 = 2$.

(5) The portions of the sides intercepted by the ellipses are semi-conjugate diameters.

(6) The tangents at the points where the ellipses cut the perpendiculars meet the sides in points which lie on lines parallel to the sides of the triangle.

(7) The common chords of the ellipses pass through the angles of the triangle.

Solutions by R. Tucker, M.A.; Mr. James Wilson; Mr. W. K. Clifford; and Alpha.



1. We have, first,

$a_1^2 = \text{AS . AD}, \; a_2^2 = \text{BS . BE}, \; a_3^2 = \text{CS . CF}.$

Now CESD may be inscribed in a circle,

$\therefore \; \text{BS . BE} = \text{BD . BC}$; similarly

$\text{CS . CF} = \text{CD . CB}$;

$\therefore \; a_2^2 + a_3^2 = \text{BD . BC} + \text{CD . BC} = a^2$;

so, $a_3^2 + a_1^2 = b^2$, and $a_1^2 + a_2^2 = c^2$;

$\therefore \; a_1^2 + a_2^2 + a_3^2 = \tfrac{1}{2}(a^2+b^2+c^2)$.

Or it may be proved as follows:—

$a_1^2 = \text{AS . AD} = bc \cos A = \tfrac{1}{2}(b^2+c^2-a^2)$;

$\therefore \; a_1^2 + a_2^2 + a_3^2 = \tfrac{1}{2}(b^2+c^2-a^2) + \tfrac{1}{2}(c^2+a^2-b^2) + \tfrac{1}{2}(a^2+b^2-c^2) = \tfrac{1}{2}(a^2+b^2+c^2)$.

2. $b_1^2 = a_1^2 - (\text{AS})^2 = \text{AS . SD}$;

and AEDB may be inscribed in a circle,

$\therefore \; \text{AS . SD} = \text{BS . SE} = \text{CS . SF}$

$= (2\text{R} \cos A)(2\text{R} \cos B \cos C)$;

$\therefore \; b_1^2 = b_2^2 = b_3^2 = 4\text{R}^2 \cos A \cos B \cos C.$

It may be otherwise proved thus:—

$b_1^2 = a_1^2 - (\text{AS})^2 = \tfrac{1}{2}(b^2+c^2-a^2) - 4\text{R}^2 \cos^2 A$

$= 2\text{R}^2(\sin^2 A + \sin^2 B + \sin^2 C - 2)$

$= 4\text{R}^2 \cos A \cos B \cos C = b_2^2 = b_3^2.$

3. $h_1^2 = \dfrac{b_1^4}{a_1^2}, \; a_1^2 = \text{AS . AD} =$

$(2\text{R} \cos A)(2\text{R} \sin B \sin C)$,

$\therefore \; h_1^2 = (4\text{R}^2 \cos A \cos B \cos C) \cot B \cot C$;

$\therefore \; h_1^2 + h_2^2 + h_3^2 = b_1^2 \, \Sigma (\cot B \cot C)$

$= b_1^2 \cot A \cot B \cot C \, \Sigma (\tan A)$

$= b_1^2 = b_2^2 = b_3^2$

$= 4\text{R}^2 \cos A \cos B \cos C.$

Otherwise, as follows:—

$$h_1^2 + h_2^2 + h_3^2 = \frac{b_1^4}{a_1^2} + \frac{b_2^4}{a_2^2} + \frac{b_3^4}{a_3^2}$$

$$= b_1^2\left(\frac{b_1^2}{a_1^2} + \frac{b_2^2}{a_2^2} + \frac{b_3^2}{a_3^2}\right)$$

$$= b_1^2\left(\frac{\text{SD}}{\text{AD}} + \frac{\text{SE}}{\text{BE}} + \frac{\text{SF}}{\text{CF}}\right)$$

$$= b_1^2\left(\frac{\Delta\,\text{BSC}}{\Delta\,\text{ABC}} + \frac{\Delta\,\text{CSA}}{\Delta\,\text{ABC}} + \frac{\Delta\,\text{ASB}}{\Delta\,\text{ABC}}\right)$$

$$= b_1^2 = b_2^2 = b_3^2.$$

4. $e_1^2 + e_2^2 + e_3^2 = \dfrac{\text{AS}}{\text{AD}} + \dfrac{\text{BS}}{\text{BE}} + \dfrac{\text{CS}}{\text{CF}} = 2.$

Or, $e_1^2 = \dfrac{2\text{R} \cos A}{c \sin B} = \dfrac{\sin 2A}{2 \sin A \sin B \sin C}$;

$$\therefore \; e_1^2 + e_2^2 + e_3^2 = \frac{\Sigma (\sin 2A)}{2 \sin A \sin B \sin C}$$

$$= \frac{4 \sin A \sin B \sin C}{2 \sin A \sin B \sin C} = 2.$$

5. $\tan \text{BAD} \tan \text{CAD} = \dfrac{\text{BD . DC}}{(\text{AD})^2}$

$$= \frac{\text{AD . DS}}{(\text{AD})^2} = \frac{\text{SD}}{\text{AD}} = \frac{b_1^2}{a_1^2},$$

which proves the property.

6. Call L the pole of AD with respect to the ellipse (C). It obviously lies in the directrix (AB), and SL is perpendicular to AD, and therefore parallel to BC. If M be the pole of the same line with respect to the ellipse (B), SM is parallel to BC, and therefore LSM is a straight line.

This proves the property; and it may also be seen to follow immediately from the proof of Question 1376, in the *Educational Times* for June.

7. This is a case of the following theorem:— "If two conics have a common focus, the common chords pass through the intersections of the two directrices."

Refer the conics to two rectangular axes through the common focus, and let $u = 0$, $v = 0$, be the equations of the two directrices; then the equations of the conics may be put under the forms

$$x^2 + y^2 = lu^2, \quad x^2 + y^2 = mv^2;$$

hence, by subtraction, the property is proved.

Again, describe the auxiliary circles of the three ellipses. If we reciprocate the system with respect to S, the only change will be that these circles will be transformed into the ellipses, and *vice versâ*. Now the *sum* of the squares of the radii of the circles (B) and (C) is BS . BE + CS . CF = BC²; hence they cut orthogonally. By reciprocation, every common tangent to the ellipses subtends a right angle at S. The *difference* of the squares of the radii of (B) and (C) is BS . BE − CS . CF = BD . BC − CD . CB = BD² − CD²; hence they cut in the perpendicular AD. Reciprocally, the common tangents to the ellipses are parallel to the sides of the triangle. Moreover, the points of contact are on the sides; since, in the reciprocal figure, the tangents at the points of intersection of the circles pass through the angular points. And since the common tangents to the circles intersect on the sides of the triangle, we learn by reciprocation that the common chords of the ellipses pass through the angular points.

8. We add some general considerations of which this last proposition is a particular case. Consider any two conics, U, V; any point (ξ, η, ζ) has two polars, ΔU, ΔV, where Δ stands for

$$\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}.$$

These meet in a point which we shall call the *polar opposite* of (ξ, η, ζ). Similarly, the line joining the poles of a right line may be called its *polar opposite*.

Now consider the equations $S=0$, $S=LM$, where L, M are common chords.

The polars are $\Delta S=0$, $\Delta S=L\Delta M+M\Delta L$.

If $\Delta L=0$, these meet on L; that is,

(*a*) If any point lie on a common chord, its polar opposite lies on the same chord.

If also $\Delta M=0$, they coincide, or

(β) The intersection of a pair of common chords has only one polar. It is easily shown that this is a line joining intersections of common tangents. It follows that

(γ) The polar opposite of any point in a straight line with two opposite intersection of common tangents, is an intersection of common chords.

Let $\Delta(L+KM)=0$, then the polars meet on $L-KM=0$. That is

(δ) Lines joining two polar opposites to an intersection of common chords, form, with the chords, an harmonic pencil.

Next, let the equations be $LM+N^2$, $LM+R^2$, so that L, M are common tangents. The polars are now

$$\left.\begin{array}{l}(L\Delta M+M\Delta L)+2N\Delta N=0\\(L\Delta M+M\Delta L)+2R\Delta R=0\end{array}\right\}$$

If $\Delta N=0$, these intersect on R, or

(ϵ) If a point lie on one chord of contact of a pair of common tangents, its polar opposite lies on the other.

If $\Delta(N+KR)=0$, the polars meet on $R+KN=0$; or

(ζ) If the locus of a point is a line through the intersection of the chords of contact of a pair of common tangents, the locus of its polar opposite is another line through the same intersection.

Thirdly, consider the case of double contact, S, $S+L^2$. Here the polars are ΔS, $\Delta S+2L\Delta L$. These always meet on L, showing that

(η) If two conics have double contact, the polar opposite of any point whatever lies on the chord of contact.

If $\Delta L=0$, they coincide, or

(θ) A point on the chord of contact has only one polar, which is also the locus of its polar opposites.

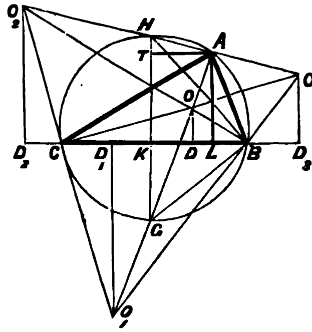
(ι) In general, if the locus of a point be a straight line, $l\xi+m\eta+n\zeta=0$, the locus of its opposite is the conic

$$\begin{vmatrix}\frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}\\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz}\\ l & m, & n\end{vmatrix}=0,$$

which we may call the *polar conic* of the line (*lmn*). As the discriminant is of the third degree in (*lmn*), it appears that the envelop of lines whose polar conics break up into two right lines is a curve of the third class.

1425 (Proposed by Mr. J. R. Wilson, Jesus College, Cambridge.)—Given, of a triangle, the base, the rectangle contained by the radii of the two escribed circles whose centres lie on the lines bisecting the angles at the base, and the rectangle contained by the sides; to construct the triangle.

Solutions by Mr. W. H. Levy; Mr. James Wilson; Mr. W. Hopps; and the Proposer.



1. *First Solution.—Construction.* Through the middle (K) of the *given* base BC draw the perpendicular GKH; and inflect BG to KG so that $BG^2:BK^2=R_1:R_2$, R_1 being the given rectangle of the *sides*, and R_2 the *given* rectangle of the radii of the escribed circles. Through the points B, G, C, describe a circle cutting GKH in H. In KH find T, so that $TK.HG=R_1$; and draw TA parallel to BC, meeting the circle in A; then will ABC be the triangle required.

Demonstration. In the bisectors of the angles B and C take O_2, O_3, the *centres*, and let O_2D_2, O_3D_3 (perpendiculars upon BC) be the *radii* of the escribed circles. Then TK = AL, the perpendicular from A upon BC; and, by construction, BC = the given base; whence, by known properties,

$$AC.AB=TK.HG=R_1 \text{ and}$$
$$=O_2D_2.O_3D_3=TK.HK;$$
$$\therefore AC.AB:O_2D_2.O_3D_3=HG:HK$$
$$=BG^2:BK^2=R_1:R_2;$$
$$\therefore O_2D_2.O_3D_3=R_2,$$

therefore ABC is the triangle required.

2. A variation of the *first* method of Solution may be obtained by inflecting BH to KH, so that

$$BH^2:BK^2=R_1:R_1-R_2,$$

then drawing a circle through B, H, C, and determining the points T, A, as in the preceding article. For, by the method of proof there given,

$$AC \cdot AB : AC \cdot AB - O_2D_2 \cdot O_3D_3 = HG : KG$$
$$= BH^2 : BK^2 = R_1 : R_1 - R_2;$$

whence $AC \cdot AB = R_1$, and $O_2D_2 \cdot O_3D_3 = R_2$; therefore ABC is the triangle required.

3. *Second Solution.* — *Construction.* To the *given* base BC add the lines $BD_2 = CD_2$, so that $BD_2 \cdot CD_2 = BD_3 \cdot CD_3 = R_2$; and divide D_2D_3 in D, so that $DD_2 \cdot DD_3 = R_1$. With centres B and C, and radii DD_3, DD_2, describe arcs intersecting in A; then will ABC be the triangle required.

Demonstration. Draw DO, D_2O_2, D_3O_3, perpendicular to BC, to meet the bisectors of the angles B and C, and bisect BC in K. By similar triangles, CD_2 (or BD_3) $: D_2O_2 = D_3O_3 : CD_3$,

$$\therefore\ BD_3 \cdot CD_3 = O_2D_2 \cdot O_3D_3 = R_2.$$

Again, $BD_2 = CD_3 = \frac{1}{2}(BC + CA + AB)$, and $KD = \frac{1}{2}(AC - AB)$,

$$\therefore\ DD_2 = AC, \text{ and } DD_3 = AB;$$

whence $DD_2 \cdot DD_3 = AC \cdot AB = R_1$.

Hence ABC is the triangle required.

4. Let O, O_1 be the centres, and OD, O_1D_1 the radii of the other two circles of contact, then

$$AC \cdot AB - O_2D_2 \cdot O_3D_3 = TK \cdot KG = AL \cdot KG$$
$$= OD \cdot O_1D_1 = BD \cdot DC.$$

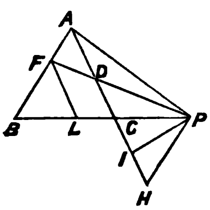
$AC \cdot AB = OD \cdot O_1D_1 + O_2D_2 \cdot O_3D_3$, and

$$BD \cdot DC : BC^2 = TK : 4HK$$
$$= AC \cdot AB : 4BH^2 \text{ (or } O_2O_3^2).$$

From this another method of construction may be easily obtained.

1426 (Proposed by Mr. James Wilson, 45th Regiment, Curragh Camp.)—Find a point in the base produced of a plane triangle, through which if a straight line be drawn cutting off from the triangle a given area, the ratio of the segments of the transversal may be given.

Solutions (1) by Mr. W. Hopps; (2) by Mr. James Wilson, and Alpha.



1. *First Solution.*—*Analysis.* Let ABC be the given triangle, P the required point in the base BC produced, and PDF the *transversal*, which cuts from the triangle ABC the triangle AFD of a *given* area, and so that $FD : DP$ = a *given* ratio. Draw PH parallel to AB, meeting AC produced in H; also draw PI perpendicular to AH, and join AP. Then, the triangle AFD being given in area, and $FD : DP$ being a given ratio, the triangle ADP is also given in area. But, by parallels, $AD : DH = FD : DP$ = a given ratio; therefore the triangle AHP is given in area; that is, $AH \cdot PI$ = a given area. But PI has a given ratio to CH, because the triangle HPC is equiangular with the given triangle ABC; therefore $AH \cdot CH$ = a given area. Hence we have to determine the point H in AC produced, so that $AH \cdot CH$ = a given area, which is a known and easy problem; and having found the point H, draw HP parallel to AB to meet BC produced, which determines the point P. Lastly, draw FL parallel to AC, then $LC : CP = FD : DP$ = a given ratio; but CP is given, therefore LC is also given; and drawing LF parallel to AC determines the point F, and consequently the position of the transversal PDF.

2. *Second Solution.*—Let $PD = m \cdot DF$..(1), $m : 1$ being the *given* ratio; then, since the area of the triangle ADF is *given*,

$$AD \cdot AF = k^2 = \textit{a given area} \ \ldots\ldots\ (2);$$

and, from the triangles BPF, CPD, cut respectively by the transversals ADC, AFB, we have

$$PC : AF = BC \cdot PD : BA \cdot FD$$
$$= m \cdot BC : BA \ \ldots\ldots\ (3),$$

$$BP : AD = BC \cdot PF : AC \cdot DF$$
$$= (m+1)\,BC : AC \ \ldots\ldots\ (4).$$

From (2), (3), (4), we obtain

$$BP \cdot PC : k^2 = m(m+1)\,BC^2 : AC^2;$$
$$\therefore\ BP \cdot PC = \textit{a given area};$$

hence the point P is determined.

1430 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—Find the *mean* distance from the centre of a sphere of all the points, (1) within its *surface*, (2) within the *circumference* of one of its great circles.

Solution by the Proposer.

1. Consider the sphere to be made up of n concentric shells of equal thickness t; then the number (N) of points within the rth shell is *proportional* to its volume, or

$$N = \tfrac{4}{3}\pi\lambda t^3\left\{r^3 - (r-1)^3\right\} = 4\pi\lambda t^3\left(r^2 - r + \tfrac{1}{3}\right) \ldots (1),$$

λ being a *constant;* and since the distance of *any point* within this shell from the centre lies between rt and $(r-1)t$, the *sum* of the distances of all the (N) points within it from the centre must lie between rtN and $(r-1)tN$; that is, between

$$4\pi\lambda t^4\left\{r^3 + F_2(r)\right\},\quad 4\pi\lambda t^4\left\{r^3 + F'_2(r)\right\} \ldots (2),$$

F_m denoting a function of the mth degree.

Giving r the values $1, 2, 3, \ldots n$, and adding the resulting values of (2), we find that the sum of the distances of all the points within these n shells from the centre must lie between

$$4\pi\lambda t^4\left\{\tfrac{1}{4}n^4 + F_3(n)\right\},\quad 4\pi\lambda t^4\left\{\tfrac{1}{4}n^4 + F'_3(n)\right\} \ldots (3).$$

If a be the radius of the sphere, $nt = a$, and thus (3) becomes

$$\pi\lambda a^4\left\{1 + 4\frac{F_3(n)}{n^4}\right\},\quad \pi\lambda a^4\left\{1 + 4\frac{F'_3(n)}{n^4}\right\} \ldots (4).$$

If now we conceive the thickness (t) of each shell to *decrease* without limit, and consequently the number (n) of these shells to *increase* without limit, we shall obtain from (4), the sum (S_1) of all the points within the sphere from its centre. For, when n *increases* without limit, $F_3(n) : n^4$ and $F'_3(n) : n^4$ both *decrease* without limit; and thus (4) gives

$$S_1 = \pi\lambda a^4 \quad\ldots\ldots\ldots\ldots\ldots(5).$$

The *mean* (or *average*) distance (M_1) of all the points from the centre is found by dividing the sum (S_1) of the distances of all these points by their number, that is, by $\frac{4}{3}\pi\lambda a^3$;

$$\therefore\ M_1 = \frac{S_1}{\frac{4}{3}\pi\lambda a^3} = \tfrac{3}{4}a \quad\ldots\ldots\ldots\ldots(6).$$

The mean distance of all the points within a *sphere* from the centre is, therefore, *three-fourths of the radius.*

2. Using the same method for the circle, that is, considering the circle to be composed of n concentric rings of equal breadth t, and finding the limit of the sum (S_2) of the distances of all the points within these rings from the centre, when t decreases and n increases without limit, the expressions of Art. 1 become

$$N = \pi\lambda t^2(2r-1) \quad\ldots\ldots\ldots\ldots(1')$$

$$2\pi\lambda t^3\left\{r^2+F_1(r)\right\},\ \ 2\pi\lambda t^3\left\{r^2+F'_1(r)\right\} \ldots(2')$$

$$2\pi\lambda t^3\left\{\tfrac{1}{3}n^3+F_2(n)\right\},\ \ 2\pi\lambda t^3\left\{\tfrac{1}{3}n^3+F'_2(n)\right\} \ldots(3')$$

$$\tfrac{2}{3}\pi\lambda a^3\left\{1+3\frac{F_2(n)}{n^3}\right\},\ \ \tfrac{2}{3}\pi\lambda a^3\left\{1+3\frac{F'_2(n)}{n^3}\right\} \ (4')$$

$$S_2 = \tfrac{2}{3}\pi\lambda a^3 \ldots\ldots\ldots\ldots(5')$$

$$M_2 = \frac{S_2}{\pi\lambda a^2} = \tfrac{2}{3}a \quad\ldots\ldots\ldots\ldots(6').$$

The mean distance of all the points within a *circle* from its centre is, therefore, *two-thirds of the radius.*

3. With the notation and processes of the *Integral Calculus*, it will be seen that the methods in Arts. 1 and 2 give

$$\tfrac{4}{3}\pi\lambda a^3 M_1 = \int_0^a r\,(4\pi\lambda r^2 dr) = \pi\lambda a^4;$$

$$\pi\lambda a^2 M_2 = \int_0^a r\,(2\pi\lambda r dr) = \tfrac{2}{3}\pi\lambda a^3;$$

$$\therefore\ M_1 = \tfrac{3}{4}a,\quad M_2 = \tfrac{2}{3}a.$$

1172 (Proposed by Mr. S. Watson.)—Find the *area-locus* of the centre of an ellipse described about a given triangle.

Note.—By a "*locus*" is generally understood a curve or straight line traced out by a moving point. Cases, however, occur in which the moving point may lie anywhere within some particular area, and may thus be said to trace out this area. It is proposed to call the area thus traced out, the *area-locus* of the point.

SOLUTION BY W. J. MILLER, B.A., MATHEMATICAL MASTER, HUDDERSFIELD COLLEGE.

1. Let the position of a point (P) be determined by its *triangular coordinates* (x, y, z), which express the ratios of the triangles PBC, PCA, PAB, to the triangle of reference ABC; then, by adapting to this system the *trilinear equations* on p. 255 of Salmon's "Conics" (4th ed.), the *circumscribed conics* which have their centres at (f, g, h) will be given by the equation

$$\frac{\lambda f}{x} + \frac{\mu g}{y} + \frac{\nu h}{z} = 0 \ldots\ldots\ldots\ldots(1);$$

where $\lambda = -f+g+h,\ \mu = f-g+h,\ \nu = f+g-h,$

or $\lambda+2f = \mu+2g = \nu+2h = f+g+h = 1.$

The required *area-locus* will be obtained by ascertaining the limits within which (f, g, h) must lie in order that the *conic* (1) may be an *ellipse.*

Now the general equation of the second degree,

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy = 0,$$

will represent an *ellipse*, if

$$(w+w'-u'-v')^2 - (u+w-2v')(v+w-2u')$$

is *negative* ("Messenger of Math.," vol. i. pp. 95, 222); hence, in order that (1) may be an *ellipse*,

$$(\lambda f+\mu g-\nu h)^2 - 4\lambda\mu fg, \text{ or}$$

$$[(f+g)^2-h^2]\,[(f-g)^2-h^2]$$

must be *negative*, or $\lambda\mu\nu$ *positive*, which will require that λ, μ, ν should be *all positive, or only one positive.*

Now, in terms of (f, g, h) as triangular coordinates, $\lambda=0$, $\mu=0$, $\nu=0$ are the equations of the straight lines which pass through the middles of the sides of the triangle of reference (that is, of the *given* triangle); hence λ, μ, ν will be *all positive* for any point *within* the triangle formed by these lines; and for any point in the three angular spaces contained between each two of the sides of this triangle, produced vertically outwards through its angles, *two* of the three λ, μ, ν will be *negative*, and the *third positive.* These four spaces, therefore, form the *area-locus* required.

2. The conics which *touch the sides* of the given triangle, and have their centres at (f, g, h) will be given by the equation

$$\sqrt{(\lambda x)}+\sqrt{(\mu y)}+\sqrt{(\nu z)} = 0, \text{ or}$$

$$\lambda^2x^2+\mu^2y^2+\nu^2z^2-2\mu\nu yz-2\nu\lambda zx-2\lambda\mu xy=0 \ \ldots(2);$$

and we shall find that $\lambda\mu\nu$ = a *positive* quantity is *also* the condition that (2) should be an *ellipse.*

The *area-locus* of the centre of (2) is, therefore, the same as that of (1); the triangle whose angular points are at the middles of the sides of the given triangle being the *area-locus* of the centre of the *inscribed* ellipses, and the three exterior angular spaces of this triangle the *area-locus* of the centres of the *escribed* ellipses.

Note.—The preceding solution is similar to that which I have given to Quest. 2000, in the "Lady's and Gentleman's Diary" for 1863, (p. 67); and on p. 68 there is a geometrical solution, with a shaded diagram, showing the *area-locus.*

1387 (Proposed by Mr. W. K. Clifford, Trinity College, Cambridge.) — Four common tangents are drawn to a circle and an ellipse passing through the centre (O) of the circle; show that OA and OB are equally inclined to the tangent at O to the ellipse.

Solution by A. Cayley, F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge.

The elegant theorem 1387, stated and proved by Mr. W. K. Clifford in the *Educational Times* for September, is included as a particular case in the known theorem, "given three conics inscribed in the same quadrilateral, the tangents from any point to these conics form a pencil in involution."

Mr. Clifford's theorem is in fact as follows: viz., Four common tangents are drawn to a circle and an ellipse which passes through the centre O of the circle; if A, B be *opposite* intersections of the tangents, then OA, OB are equally inclined to the tangent at O to the ellipse.

This comes to saying that the tangent at O to the ellipse, say OT, is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ drawn from O to the circular points at infinity; and if we replace the circle by an arbitrary conic S, and the line infinity by an arbitrary line IJ, the theorem will be as follows.

Consider a conic S; a line meeting this conic in the points I, J; and the point O, the intersection of the tangents at I, J, or (what is the same thing) the pole of the line IJ in regard to the conic. If through the point O there be drawn any other conic Θ, and if A, B be *opposite* intersections of the common tangents of the conics S, Θ; then the tangent OT at the point O to the conic Θ is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ; or, as we may also express it, the lines OT, OT, the lines OA, OB, and the lines OI, OJ, form a pencil in involution.

Now considering the two points or point-pair (A, B) as a conic inscribed in the quadrilateral formed by the common tangents of the conics S and Θ, the conics S and Θ and the point-pair (A, B) are a system of three conics inscribed in the same quadrilateral; and hence, by the general theorem above referred to, if O′ be any point whatever, the tangents from O′ to the conic S, the tangents from O′ to the conic Θ, and the tangents from O′ to the point-pair (that is, the two lines O′A, O′B) form a pencil in involution. But if O′ coincide with O, then the tangents to the conic S are the lines OI, OJ; and the tangents to the conic Θ are the coincident lines OT, OT; and we have thence the theorem in question; viz., that the lines OT, OT, the lines OI, OJ, and the lines OA, OB form a pencil in involution.

1409 (Proposed by Mr. W. K. Clifford.)— For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary.

Solution by the Proposer.

Take the point A for origin, and the rectangular tangential equation used in Question 1387, but in the more convenient form

$$(\xi-a)^2=4b\,(\eta-c)\quad\ldots\ldots\ldots\ldots\ldots(1),$$

which is evidently equivalent to the one there given. The line BC is represented by

$$\xi-a=0=\eta-b-c\quad\ldots\ldots\ldots\ldots\ldots(2);$$

it always passes through the pole of the normal, and is in fact the polar of the point of intersection of chords subtending a right angle at A.

If we assume for the general equation of a point on the curve

$$\xi-a=m\,(\eta-c)+\beta\quad\ldots\ldots\ldots\ldots\ldots(3),$$

then the equation

$$\left\{m\,(\eta-c)+\beta\right\}^2=4b\,(\eta-c)$$

must have equal roots for η, which gives $\beta=\dfrac{b}{m}$.

We shall call this the point m.

Let the intersection of BD, CE, be the point m_1, and the points D, E, m_2, m_3. The equations of B, C, will therefore be

$$\left(m_2+\frac{1}{m_2}\right)\left\{(\xi-a)-m_1\,(\eta-b-c)\right\}$$
$$=\left(m_1+\frac{1}{m_1}\right)\left\{(\xi-a)-m_2\,(\eta-b-c)\right\}\quad\ldots(4)$$
$$\left(m_3+\frac{1}{m_3}\right)\left\{(\xi-a)-m_1\,(\eta-b-c)\right\}$$
$$=\left(m_1+\frac{1}{m_1}\right)\left\{(\xi-a)-m_3\,(\eta-b-c)\right\}\quad\ldots(5).$$

These equations may be easily verified.

The angle between AB and the axis of ξ is therefore

$$\tan^{-1}\frac{m_2\left(m_1+\dfrac{1}{m_1}\right)-m_1\left(m_2+\dfrac{1}{m_2}\right)}{\left(m_2+\dfrac{1}{m_2}\right)-\left(m_1+\dfrac{1}{m_1}\right)}$$
$$=\tan^{-1}\frac{m_1+m_2}{m_1m_2-1}.$$

Hence the angle BAC is

$$\tan^{-1}\frac{m_1+m_2}{1-m_1m_2}-\tan^{-1}\frac{m_1+m_3}{1-m_1m_3}.$$

That is, its tangent is equal to that of the angle between the lines joining the points m_2, m_3 to the origin. For the latter angle is clearly

$$\tan^{-1}\frac{m_2-m_3}{1+m_2m_3}.$$

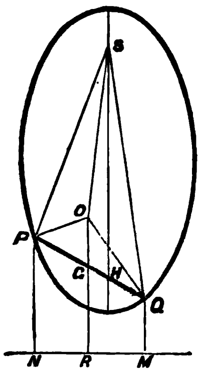
Since, therefore, the angles BAC, DAE have their tangents equal, they are either equal or supplementary.

If the conic be a circle, it is easily seen that the line BC is always at an infinite distance.

1414 (Proposed by Mr. J. R. Wilson, Jesus College, Cambridge.)—In the upper focus of a smooth ellipse, fixed with its transverse axis vertical, there is a repulsive centre of force varying as the inverse square of the distance; and a uniform heavy rod of length c is placed within it; show that, if θ be the inclination of the rod to the vertical when in equilibrium, l ($<c$) the latus rectum of the ellipse, and e its eccentricity,

$$\cos\theta = \frac{1}{e}\sqrt{\left(1-\frac{l}{c}\right)}.$$

Solution by the PROPOSER.



Let S, H be the foci of the ellipse, PQ the rod in equilibrium. The forces acting in PQ are, its weight vertically through G its middle point, the resultant repulsive force through S bisecting the angle PSQ, and the normal reactions PO, QO. Suppose the centre of force removed from S; then it may be shown that if the rod be in equilibrium under its weight and the normal reactions, it must, if not horizontal, pass through the focus H; for, if PN, GR, QM be perpendicular to the directrix, GR must be a minimum; hence (PN + QM), and therefore (HP + HQ), must be a minimum, which will be the case when PQ passes through H. This proves, mechanically, that if PO, QO be normals at the ends of a focal chord, then OG, drawn parallel to the axis, will bisect PQ. Now suppose the centre of force replaced; the resultant repulsive force will pass through O, since PO, QO the bisectors of the angles P, Q of the triangle SPQ meet in O: thus all the forces acting on PQ pass through O, and it is plain that, since the resultant of the rod's weight and the repulsive force lies between OP, OQ, the normal reactions at P, Q will so adapt themselves as to constitute an equilibrium. We have then

$$c = \text{HP}+\text{HQ} = \frac{l}{1-e^2\cos^2\theta};$$

$$\therefore\ \cos\theta = \frac{1}{e}\sqrt{\left(1-\frac{l}{c}\right)}.$$

1422 (Proposed by Matthew Collins, B.A. Senior Moderator in Mathematics and Physics Trinity College, Dublin.)—If the abscissa (x) of any point in the circumference of a circle be perpendicular to the ordinate (y) of that point, and s be the arc of the circle between the point (x, y) and a fixed point; it is required to prove that, if dx be constant,

$$\frac{dy\,ds^2}{dx\,d^2s} = a \text{ constant} \quad\ldots\ldots\ldots(1).$$

$$\frac{dy\,d^3y}{(d^2s)^2} = \frac{ds^2\,d^3y}{dy\,(d^2y)^2} = a \text{ constant} \quad\ldots\ldots\ldots(2).$$

$$\frac{d^2y}{dy}+\frac{d^4y}{d^3y} = 2\frac{d^3s}{d^2s} \quad\ldots\ldots\ldots\ldots(3).$$

$$\frac{d^2y}{dy}+2\frac{d^3y}{d^2y}-\frac{d^4y}{d^3y} = 2\frac{d^2s}{ds} \quad\ldots\ldots\ldots\ldots(4.)$$

Solution by ALPHA; Mr. J. WILSON; and R. TUCKER, M.A.

Let a be the radius of the circle; then, taking he *centre* as origin,

$$y^2 = a^2-x^2, \text{ and } ds^2 = dx^2+dy^2;$$

hence, dx being *constant*, we have

$$dy = -xy^{-1}dx;\ \ d^2y = -a^2y^{-3}dx^2;$$

$$d^3y = -3a^2xy^{-5}dx^3;\ \ d^4y = -3a^2y^{-7}(a^2+4x^2)\,dx^4;$$

$$ds = ay^{-1}dx;\ \ d^2s = axy^{-3}dx^2;$$

$$d^3s = ay^{-5}(a^2+2x^2)\,dx^3;$$

and from these values we readily obtain

$$\frac{dy\,ds^2}{dx\,d^2s} = a = a \text{ constant} \quad\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{dy\,d^3y}{(d^2s)^2} = \frac{ds^2\,d^3y}{dy\,(d^2y)^2} = 3 = a \text{ constant} \quad\ldots\ldots(2).$$

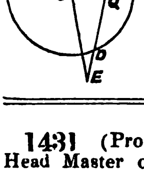
$$\frac{d^2y}{dy}+\frac{d^4y}{d^3y} = \frac{2(a^2+2x^2)}{xy^2} = 2\frac{d^3s}{d^2s} \quad\ldots\ldots\ldots\ldots(3).$$

$$\frac{d^2y}{dy}+\frac{2d^3y}{d^2y}-\frac{d^4y}{d^3y} = \frac{2xdx}{y^2} = -\frac{2dy}{y} = \frac{2d^2s}{ds} \quad\ldots(4).$$

If the origin be *anywhere*, so that the equation of the circle is of the form $(x-h)^2+(y-k)^2 = a^2$, we shall have to put $x-h$ for x, and $y-k$ for y; and the foregoing relations obviously remain unchanged.

1424 (Proposed by T. T. Wilkinson, F.R.A.S., Grammar School, Burnley.)—Having given the points P, E, and the line OE by position in a vertical plane; it is required to find the point O, in the line OE, so that if a circle be described with the centre O and a given radius, a heavy body let fall from P along PE may descend through the chord CD (cut off from PE by the circle) in a given time.

Solution by Mr. J. WILSON; and the PROPOSER.

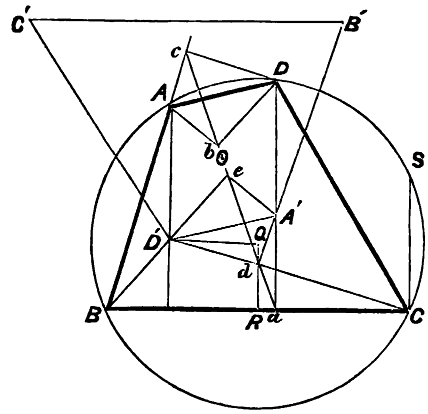


Let c be the cosine of the inclination of PE to the vertical; then CD ($=\frac{1}{2}gct^2$, t being the *given* time of descending through CD) is given: hence the perpendicular distance (OQ) of O from PE is *given*, which determines the point O.

1431 (Proposed by Thomas Dobson, B.A., Head Master of the Royal Grammar School, Hexham.)—Of four given points in a circle it is

known that every three may be taken as the intersections of tangents to a parabola, of which the fourth is the focus. Prove that the tangents at the vertices of the parabolas thus described intersect in a point, such that the sum of the squares of its distances from the four given points is equal to the square of the diameter of the circle.

Solution by the PROPOSER.



Let ABCD be a quadrilateral figure in a circle of which Q is the centre; A′B′C′D′ the intersections of the perpendiculars of the triangles BDC, ACD, ABD, and ABC respectively. Draw Da, QR perpendicular to BC.

It is known that the circle through the feet of the perpendiculars of the triangle ABC passes through R, bisects AD′, and has its centre at the middle point of QD′. Hence AD′ = 2QR = A′D; and ADA′D′ is a parallelogram. Similarly A′C is is equal and parallel to AC′, and so on; and ABA′B′, ACA′C′, BCB′C′, BDB′D′, CDC′D′ are parallelograms of which the diagonals all pass through O, the intersection of AA′ and DD′. For DD′ is one diagonal of CDC′D′, therefore the other diagonal CC′ passes through its middle point O; and so on.

The figure A′B′C′D′ is evidently identical with ABCD.

From any angular point of ABCD, as D, draw perpendiculars Da, Db, Dc, to t e lin s BC, CA, AB; and from the corresponding point A′ draw perpendiculars A′a, A′d, A′e to CB, CD′, BD′, then $abcde$ O will be a straight line. For, since the angles BA′C, BD′C are the supplements of the equal angles BDC, BAC, the points B, D′, A′, C are in the arc of the circle about the triangle BD′C; and it is known that abc, ade are straight lines. Now the feet of the perpendiculars from B and C on CA, CD, BA, BD lie in a circle on diameter BC; and a circle may be described through each group of points adA′C, abDC, and A′dD′e,

$\therefore \angle$ A′ad = A′Cd = ABD = bCD = baD;

hence adb is a straight line, and therefore $abcde$ is a straight line. But the triangles ADc, A′D′d are identically equal, and their equal sides parallel, therefore DcD′d is a parallelogram, of which one diagonal DD′ is bisected in O, therefore the other diagonal cd, that is $abcde$, passes through O.

Similarly, if perpendiculars be drawn from A on the sides of the triangle BCD, and from D′ on those of BA′C, their feet will lie in a straight line through O; and this holds also when perpendiculars are drawn to corresponding lines from B and C.

But the line abc is the tangent at the vertex of the parabola of which D is the focus, and to which AB, BC, CA are tangents; and the straight lines through O, which are the loci of the feet of perpendiculars from A, B, C, respectively, are the tangents at the vertices of the three other cognate parabolas. Hence the tangents at the vertices of the four parabolas pass through the point O.

Since D and D′ are equally distant from abc, it follows that the directrix corresponding to the focus D passes through D′, &c.

Now AD′ = 2QR, and if BQ be produced to meet the perpendicular from C in S, CS = 2QR = AD′, and BS = the diameter (D).

$$\text{Also } AD'^2 = D^2 - BC^2;$$
$$\text{and } 2AO^2 + 2DO^2 = AD^2 + AD'^2;$$
$$\therefore\ 2AO^2 + 2DO^2 = D^2 - BC^2 + AD^2;$$

this, with three corresponding expressions involving BO, CO, gives, by adding,

$$AO^2 + BO^2 + CO^2 + DO^2 = D^2.$$

1435 (Proposed by the Editor.) – Show how to find the area of the Pedal of a curve for any origin, when the area of the Pedal for any other origin is known; and hence prove that the locus of the origin of a Pedal of constant area is a conic, and that all such loci constitute a system of similar, similarly placed, and concentric conics, whose common centre is the origin of the Pedal of least area.

Solution by the PROPOSER.

1. Let (C) represent the primitive curve, (P) the Pedal whose origin (A) has the coordinates (x, y), and (P_0) the Pedal whose origin O coincides with that of the coordinate axes. The curve (C) may be regarded as dividing the plane into two parts, distinguishable as external and internal; let α and β then be the angles, each positive and less than π, between the positive directions of the coordinate axes and that of the normal at any point M of (C), this normal being always supposed to be drawn from the curve into the external part of the plane. Further, let p and p_0 be the perpendiculars let fall respectively from the point A, and from the origin O upon the tangent at M, so that their feet m and m_0 are the points on the Pedals (P) and (P_0) which correspond to M on the primitive. The direction-angles of each perpendicular will be (α, β), or ($\pi-\alpha$, $\pi-\beta$), according as its direction coincides with, or is opposed to, that of the normal; so that if we regard p and p_0 as positive or negative ac-

cording as the one or the other of these circumstances occurs, we shall have, generally,

$$p = p_0 - x\cos\alpha - y\sin\alpha.$$

If we further denote by $d\theta$ the arc of the unit-circle, around the origin, intercepted between radii whose directions coincide with those of the normals at the extremities of the element ds of the primitive arc at M, and agree to consider the parallel elements ds and $d\theta$ as alike or unlike in sign according as their directions coincide with or are opposed to each other, the corresponding elements dP and dP_0 of the areas of the Pedals (P) and (P_0) will be

$$dP = \tfrac{1}{2}p^2 d\theta, \quad dP_0 = \tfrac{1}{2}p_0^2 d\theta,$$

and, by the preceding relation, we shall have

$$2dP = (p_0 - x\cos\alpha - y\sin\alpha)^2 d\theta;$$

whence, by integration, we deduce the equation

$$\begin{aligned} P = P_0 - A_1 x - A_2 y \\ + \tfrac{1}{2}(A_{11}x^2 + 2A_{12}xy + A_{22}y^2).\ ..(A) \end{aligned}$$

wherein P and P_0 denote the areas of the two Pedals, and the coefficients have the values

$$A_1 = \int p_0\, d\theta \cos\alpha, \quad A_2 = \int p_0\, d\theta \cos\beta,$$

$$A_{11} = \int d\theta \cos^2\alpha, \quad A_{12} = \int d\theta \cos\alpha\cos\beta,$$

$$A_{22} = \int d\theta \cos^2\beta,$$

dependent only on the position of the origin O, and on the curvature of the primitive curve. The integration in each case is, of course, to be extended to all points of the primitive curve.

2. The above formula, by means of which the area of any Pedal (P) may be found when the area of any other (P_0) is known, shows at once that the locus (A) of the origin A of a Pedal of constant area is a conic, and that all such loci constitute a system of similar, similarly placed, and concentric conics, the common centre of the loci being the point at which the integrals A_1, A_2 vanish. If we suppose the origin of our coordinate axes to coincide with this point, the equation of the locus (A) may be written

$$P = P_0 + \tfrac{1}{2}\int (x\cos\alpha + y\cos\beta)^2\, d\theta,$$

whence we learn that the common centre of all the quadric loci (A) is the origin of the Pedal of least area.

3. Let us consider, in the next place, the Pedals of a primitive arc containing a point of inflexion and having parallel normals at its extremities. The normals along such an arc will consist of pairs of like-directed parallels; but in passing from one extremity to the other the sign of $d\theta$ will change, so that the integrals A_{11}, A_{12}, A_{22}, will each consist of equal and opposite elements and vanish in consequence. In this case, the locus (A) of equal Pedal origins coincides with the straight line

$$P = P_0 - A_1 x - A_2 y.$$

4. If the primitive be a closed curve, but otherwise perfectly arbitrary, we may always conceive it to consist of arcs (C′) of the kind considered in Art. 3, and of other arcs (C″) the directions of whose normals represent exactly all possible directions round a point. But it has already been shown that for every arc (C′) the integrals A_{11}, A_{12}, A_{22} vanish, and it is easy to see that, extended over the arcs (C″), these integrals have the values

$$A_{11} = A_{12} = n\pi, \quad A_{22} = 0,$$

where n represents the number of such arcs, in other words, the number of convolutions of the primitive curve. In this case, therefore, the equation of Art. 2 becomes

$$P = P_0 + \tfrac{1}{2}n\pi(x^2+y^2) = P_0 + \tfrac{1}{2}n\pi r^2,$$

and for constant values of P represents a circle around the origin of the least Pedal.

5. In order to illustrate by an example what is meant by the area of a Pedal, let us consider the case of an ellipse with the semi-axes a, b. The focal Pedal, as is well known, is a circle whose diameter is the major axis; so that, putting for P, n, r^2 the values πa^2, 1, a^2-b^2 respectively, we find, for the area of the central Pedal, the value

$$P_0 = \tfrac{1}{2}\pi(a^2+b^2)$$

equal to the area of the semicircle whose radius is the line joining the extremities of the axes; and the area of any other Pedal is

$$P = \tfrac{1}{2}\pi(a^2+b^2+r^2).$$

For the circle $a=b$, we have

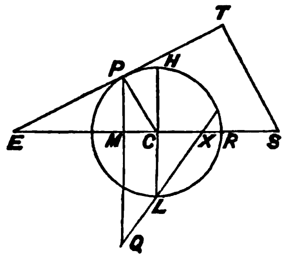
$$P = \pi a^2 + \tfrac{1}{2}\pi r^2,$$

which clearly represents the *sum* of the areas of the two loops of which the Pedal consists when its origin is without the circle. When a vanishes, the Pedal is well known to be the circle on r as diameter. Our last formula shows, however, that we must conceive this circle to be doubled. A glance at the expressions for p and dP in Art. 1 explains this distinctive feature of Pedal areas. It will be there seen that the sign of the increment dP does not depend upon that of p, which latter changes according as the Pedal origin lies on one or the other side of the tangent. We in fact consider the area of a Pedal to be the space swept by the perpendicular as the point of contact of the tangent describes the primitive arc.

[We have taken the foregoing investigation from a very interesting paper by Dr. Hirst ("On the volumes of Pedal Surfaces") recently published in the Philosophical Transactions.]

1437 (Proposed by Mr. A. Renshaw.)—If S be any point *within* or *without* the circle whose centre is C and radius CR, and a point X be taken in the diameter (CS) through S, so that CS . CX $= CR^2$; also if HCL be a diameter at right angles to CS, and LX be joined; then, if from *any* point P in the circumference, PMQ be drawn at right angles to CS, meeting CS in M and LX in Q, the perpendicular from S upon the tangent at P will always be equal to MQ.

Solutions by R. TUCKER, M.A.; Mr. J. WILSON; and the PROPOSER.



1. *First Solution.*—Let the perpendicular from S meet the tangent at P in T, and let TP cut SC in E;

$$\text{then } CE \, . \, CM. = CP^2 = CX \, . \, CS;$$
$$\therefore \; CE : CS = CX : CM,$$
$$\text{and } CE : SE = CX : MX;$$
$$\text{but } CE : SE = CP : ST,$$
$$\text{and } CX : MX = CL : MQ,$$
$$\therefore \; CP : ST = CL : MQ;$$
$$\text{and } CP = CL, \text{ therefore } ST = MQ.$$

2. *Second Solution.*— Let $x^2+y^2 = a^2$ be the equation of the circle, and (c, o), $(-x', y')$ the coordinates of S, P; then the length of the perpendicular (ST) from (c, o) on the tangent at P is

$$ST = \frac{cx'+a^2}{a};$$

$$\text{also, } MQ = a\left\{\frac{\frac{a^2}{c}+x'}{\frac{a^2}{c}}\right\} = \frac{cx'+a^2}{a};$$

$$\therefore \; ST = MQ.$$

1439 (Proposed by Professor SYLVESTER, F.R.S., Royal Military Academy, Woolwich.)— Prove that the tangents (PT, QT) from any two points (P, Q) of a parabola, as mutually limited by each other, are as their distances from the focus.

Solutions (1) by Professor SYLVESTER; (2) by Mr. JAMES WILSON; Mr. WILLIAM HOGG; Dr. RUTHERFORD, F.R.A.S.; and ALPHA; (3) by R. TUCKER, M.A.

[See Figure to next Question.]

1. *First Solution.*—The motion of a point in a parabola whose focus is F, under the action of a constant force, may be regarded as the limiting case of the motion in an ellipse whose foci are F and H, infinitely remote from F, under the action of a force varying inversely as the square of the distance from H. Hence, by the doctrine of central forces, the velocity at any point is *inversely* as the focal distance of its tangent from H, that is *directly* as the focal distance from F. But it is shown in Art. 7 of the Solution of Question 1402 (see p. 20), also by mechanical reasoning, that the velocities at any two points of the parabola, regarded as a trajectory under a constant force, are in the proportion of the segments of the tangents at these points, as limited by each other; hence these latter are as their distances from the focus.

2. *Second Solution.*—From the focus F draw FM, FN perpendicular to the tangents PT, QT. Then, by what has been proved in Arts. 4, 6 of the Solution of Quest. 1402, (see also Drew's Conics, Art. 13, or Davies's Hutton, vol. ii., p. 114,) the triangles TPF, TQF are *similar*, as are also the triangles MPF, FTN;

$$\text{therefore } TP : PF = TQ : TF;$$
$$\text{also } PF : FM = TF : FN;$$
$$\text{therefore } TP : TQ = FM : FN,$$

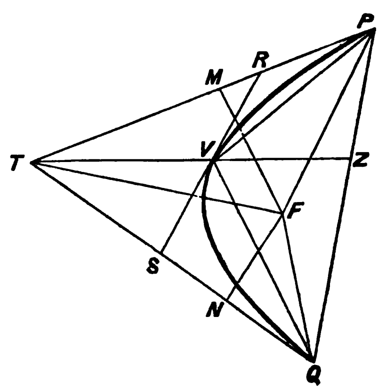
which proves the theorem.

3. Or, again,

$$\triangle TPF : TQF = TP^2 : TQ^2$$
$$= TP \, . \, FM : TQ \, . \, FN,$$
$$\therefore \; TP : TQ = FM : FN.$$

1440 (Proposed by W. J. MILLER, B.A., Mathematical Master, Huddersfield College.) — If from the intersection of two tangents to a parabola, a straight line be drawn through the point of contact of a third tangent; prove that this straight line will divide the chord of contact of the first two tangents into parts which have to one another the *duplicate ratio* of the segments of the third tangent, intercepted between its point of contact and the first two tangents.

Solutions (1) by the PROPOSER; (2) by R. TUCKER, M.A.; (3) by Mr. W. HOGG, Mathematical Master, Royal Naval School, New Cross; Mr. J. WILSON; and ALPHA.



1. *First Solution.* — It is a well known and general property of conics that four fixed tangents cut any fifth tangent in points whose anharmonic

ratio is constant. Now a parabola has one tangent at infinity; and thus the intercepts of a variable tangent, made by three fixed tangents to a parabola, have to one another a constant ratio.

By supposing the variable tangent to coincide successively with the three fixed tangents PRT, RVS, TSQ, we obtain the theorem

$$PR : RT = RV : VS = TS : SQ,$$

which was known to the Greek geometers, and is in fact given in the *Conics* of Apollonius.

The preceding proof is due to Dr. Salmon (*Conics*, 4th ed., art. 327); and a different proof may be seen in Davies's Hutton (vol. ii. p. 121).

Let TV meet the chord (PQ) of contact in Z; then, by the theorem proved above, it follows that the triangles PVR, RVT, TVS, SVQ are continual proportionals, each having to the preceding the constant ratio RV : VS;

$$\therefore \Delta PVT : RTS = PVR : RVT = RV : VS;$$
$$\Delta RTS : TVQ = RVT : TVS = RV : VS;$$
$$\therefore \Delta PVT : TVQ = PZ : ZQ = RV^2 : VS^2,$$

which proves the theorem

2. Or again,

$$\frac{PZ}{ZQ} = \frac{\Delta PVT}{\Delta TVQ} = \frac{\frac{PT}{RT}\Delta RVT}{\frac{QT}{ST}\Delta TVS}$$
$$= \frac{RS}{VS}\cdot\frac{RV}{RS}\cdot\frac{RV}{VS} = \frac{RV^2}{VS^2}.$$

3. *Second Solution.*—Taking the tangents TQ, TP as axes, let the points P, Q, V, Z be (o, c), (a, o), (f, g), (h, k); then the respective equations of the parabola PVQ, the third tangent RVS, the chord of contact PZQ, and the secant TVZ, are

$$\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{c}\right)} = 1 \ldots\ldots\ldots\ldots(1)$$
$$\frac{x}{\sqrt{(af)}} + \frac{y}{\sqrt{(cg)}} = 1 \ldots\ldots\ldots\ldots(2)$$
$$\frac{x}{a} + \frac{y}{c} = 1 \ldots\ldots\ldots\ldots(3)$$
$$my = nx \ldots\ldots\ldots\ldots(4)$$

From (1), (4), and (3), (4) we have

$$\sqrt{g} = \frac{\sqrt{(nac)}}{\sqrt{(na)}+\sqrt{(mc)}}, \quad k = \frac{nac}{na+mc};$$

also (2) gives $TR = \sqrt{(cg)}$;

$$\therefore \frac{PZ}{QZ} = \frac{c-k}{k} = \frac{mc}{na};$$
$$\frac{RV}{VS} = \frac{\sqrt{(cg)}-g}{g} = \sqrt{\left(\frac{mc}{na}\right)};$$
$$\therefore PZ : ZQ = RV^2 : VS^2.$$

1401 (Proposed by the Editor.)—Eliminate x between the two equations

$$x^5 + px^2 + qx + r = 0,$$
$$x^4 + dx^3 + cx^2 + bx + a = y;$$

and exhibit the result in the form

$$y^5 + Ay^4 + By^3 + Cy^2 + Dy + E = 0,$$

giving the values of A, B, C, D, E, in terms of p, q, r, a, b, c, d.

Solution by the Rev. Robert Harley, F.R.S.; and similarly by Mr. Samuel Bills.

Taking the Question more generally, let it be proposed to eliminate x between the equations

$$x^n + lx^{n-1} + mx^{n-2} + px^{n-3} + qx^{n-4} + \&c. = 0,$$
$$\text{and } y = a + bx + cx^2 + dx^3 + x^4 \;(= \Sigma a, \text{ say}).$$

Then it is known that the resulting equation in y will be of the form

$$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3} + Dy^{n-4} + \&c. = 0;$$

and since the coefficients of an equation become known when the sums of the powers of its roots are known, it will be sufficient for our present purpose to show how the sums of the powers of y may be calculated. Now, for every value of y there is a corresponding value of x; hence the passage from y^k to Σy^k will be effected by simply developing $(\Sigma a)^k$ in ascending powers of x, and prefixing, in the result, Σ to each of these powers. Thus, observing that $\Sigma x^0 = n$, we have

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 + d\Sigma x^3 + \Sigma x^4;$$

and since

$$\begin{aligned} y^2 = (\Sigma a)^2 &= \Sigma a^2 + 2\Sigma a \,.\, bx \\ &= a^2x^0 + 2abx + (2ac+b^2)x^2 + (2ad+2bc)x^3 \\ &\quad + (2a+2bd+c^2)x^4 + (2b+2cd)x^5 \\ &\quad + (2c+d^2)x^6 + 2dx^7 + x^8, \end{aligned}$$

$$\begin{aligned} \therefore \Sigma y^2 = na^2 &+ 2ab\Sigma x + (2ac+b^2)\Sigma x^2 \\ &+ (2ad+2bc)\Sigma x^3 + (2a+2bd+c^2)\Sigma x^4 \\ &+ (2b+2cd)\Sigma x^5 + (2c+d^2)\Sigma x^6 + 2d\Sigma x^7 \\ &+ \Sigma x^8. \end{aligned}$$

Again,

$$y^3 = (\Sigma a)^3 = \Sigma a^3 + 3\Sigma a^2 \,.\, bx + 6\Sigma a \,.\, bx \,.\, cx^2,$$

$$\begin{aligned} y^4 = (\Sigma a)^4 &= \Sigma a^4 + 4\Sigma a^3 \,.\, bx + 6\Sigma a^2 \,.\, (bx)^2 \\ &\quad + 12\Sigma a^2 \,.\, bx \,.\, cx^2 + 24\Sigma a \,.\, bx \,.\, cx^2 \,.\, dx^3, \end{aligned}$$

$$\begin{aligned} y^5 = (\Sigma a)^5 &= \Sigma a^5 + 5\Sigma a^4 \,.\, bx + 10\Sigma a^3 \,.\, bx^2 \\ &\quad + 20\Sigma a^3 \,.\, bx \,.\, cx^2 + 30\Sigma a^2 \,.\, (bx)^2 \,.\, cx^2 \\ &\quad + 60\Sigma a^2 \,.\, bx \,.\, cx^2 \,.\, dx^3 \\ &\quad + 120a \,.\, bx \,.\, cx^2 \,.\, dx^3 \,.\, x^4. \end{aligned}$$

And similarly for higher powers of y. Developing the right-hand members, and taking the sums of the powers of x and y as before, we obtain the following results :—

$$\begin{aligned} \Sigma y^3 = na^3 &+ 3a^2b\Sigma x + (3a^2c + 3ab^2)\Sigma x^2 \\ &+ (3a^2d + 6abc + b^3)\Sigma x^3 \\ &+ (3a^2 + 6abd + 3ac^2 + 3b^2c)\Sigma x^4 \\ &+ (6ab + 6acd + 3b^2d + 3bc^2)\Sigma x^5 \\ &+ (6ac + 3ad^2 + 3b^2 + 6bcd + c^3)\Sigma x^6 \\ &+ (6ad + 6bc + 3bd^2 + 3c^2d)\Sigma x^7 \\ &+ (3a + 6bd + 3c^2 + 3cd^2)\Sigma x^8 \\ &+ (3b + 6cd + d^3)\Sigma x^9 + (3c + 3d^2)\Sigma x^{10} \\ &+ 3d\Sigma x^{11} + \Sigma x^{12} \end{aligned}$$

$$
\begin{aligned}
\Sigma y^4 = na^4 &+ 4a^3b\Sigma x + (4a^3c + 6a^2b^2)\,\Sigma x^2 \\
&+ (4a^3d + 12a^2bc + 4ab^3)\,\Sigma x^3 \\
&+ (4a^3 + 12a^2bd + 6a^2c^2 + 12ab^2c + b^4)\,\Sigma x^4 \\
&+ (12a^2b + 12a^2cd + 12ab^2d + 12abc^2 \\
&\qquad + 4b^3c)\,\Sigma x^5 \\
&+ (12a^2c + 6a^2d^2 + 12ab^2 + 24abcd + 4ac^3 \\
&\qquad + 4b^3d + 6b^2c^2)\,\Sigma x^6 \\
&+ (12a^2d + 24abc + 12abd^2 + 12ac^2d + 4b^3 \\
&\qquad + 12b^2cd + 4bc^3)\,\Sigma x^7 \\
&+ (6a^2 + 24abd + 12ac^2 + 12acd^2 + 12b^2c \\
&\qquad + 6b^2d^2 + 12bc^2d + c^4)\,\Sigma x^8 \\
&+ (12ab + 24acd + 4ad^3 + 12b^2d + 12bc^2 \\
&\qquad + 12bcd^2 + 4c^3d)\,\Sigma x^9 \\
&+ (12ac + 12ad^2 + 6b^2 + 24bcd + 4bd^3 + 4c^3 \\
&\qquad + 6c^2d^2)\,\Sigma x^{10} \\
&+ (12ad + 12bc + 12bd^2 + 12c^2d + 4cd^3)\,\Sigma x^{11} \\
&+ (4a + 12bd + 6c^2 + 12cd^2 + d^4)\,\Sigma x^{12} \\
&+ (4b + 12cd + 4d^3)\,\Sigma x^{13} + (4c + 6d^2)\,\Sigma x^{14} \\
&+ 4d\Sigma x^{15} + \Sigma x^{16},
\end{aligned}
$$

$$
\begin{aligned}
y^5 = na^5 &+ 5a^4b\Sigma x + (5a^4c + 10a^3b^2)\,\Sigma x^2 \\
&+ (5a^4d + 20a^3bc + 10a^2b^3)\,\Sigma x^3 \\
&+ (5a^4 + 20a^3bd + 10a^3c^2 + 30a^2b^2c + 5ab^4)\,\Sigma x^4 \\
&+ (20a^3b + 20a^3cd + 30a^2b^2d + 30a^2bc^2 + 20ab^3c \\
&\qquad + b^5)\,\Sigma x^5 \\
&+ (20a^3c + 10a^3d^2 + 30a^2b^2 + 60a^2bcd + 10a^2c^3 \\
&\qquad + 20ab^3d + 30ab^2c^2 + 5b^4c)\,\Sigma x^6 \\
&+ (20a^3d + 60a^2bc + 30a^2bd^2 + 30a^2c^2d + 20ab^3 \\
&\qquad + 60ab^2cd + 20abc^3 + 10b^3c^2 + 5b^4d)\,\Sigma x^7 \\
&+ (10a^3 + 60a^2bd + 30a^2c^2 + 30a^2cd^2 + 60ab^2c \\
&\qquad + 30ab^2d^2 + 60abc^2d + 5ac^4 + 20b^3cd \\
&\qquad + 10b^2c^3 + 5b^4)\,\Sigma x^8 \\
&+ (30a^2b + 60a^2cd + 10a^2d^3 + 60ab^2d + 60abc^2 \\
&\qquad + 60abcd^2 + 20ac^3d + 20b^3c + 10b^3d^2 \\
&\qquad + 30b^2c^2d + 5bc^4)\,\Sigma x^9 \\
&+ (30a^2c + 30a^2d^2 + 30ab^2 + 120abcd + 20abd^3 \\
&\qquad + 20ac^3 + 30ac^2d^2 + 20b^3d + 30b^2c^2 \\
&\qquad + 30b^2cd^2 + 20bc^3d + c^5)\,\Sigma x^{10} \\
&+ (30a^2d + 60abc + 60abd^2 + 60ac^2d + 20acd^3 \\
&\qquad + 10b^3 + 60b^2cd + 10b^2d^3 + 20bc^3 + 30bc^2d^2 \\
&\qquad + 5c^4d)\,\Sigma x^{11} \\
&+ (10a^2 + 60abd + 30ac^2 + 60acd^2 + 5ad^4 \\
&\qquad + 30b^2c + 30b^2d^2 + 60bc^2d + 20bcd^3 + 5c^4 \\
&\qquad + 10c^3d^2)\,\Sigma x^{12} \\
&+ (20ab + 60acd + 20ad^3 + 30b^2d + 30bc^2 \\
&\qquad + 60\ cd^2 + 5bd^4 + 20c^3d + 10c^2d^3)\,\Sigma x^{13} \\
&+ (20ac + 30ad^2 + 10b^2 + 60bcd + 20bd^3 + 10c^3 \\
&\qquad + 30c^2d^2 + 5cd^4)\,\Sigma x^{14} \\
&+ (20ad + 20bc + 30bd^2 + 30c^2d + 20cd^3 \\
&\qquad + d^5)\,\Sigma x^{15} \\
&+ (5a + 20bd + 10c^2 + 30cd^2 + 5d^4)\,\Sigma x^{16} \\
&+ (5b + 20cd + 10d^3)\,\Sigma x^{17} + (5c + 10d^2)\,\Sigma x^{18} \\
&+ 5d\Sigma x^{19} + \Sigma x^{20}.
\end{aligned}
$$

The coefficients of the equation in y are connected with the sums of the powers of its roots by the relations

$$
\begin{aligned}
-\mathrm{A} &= \Sigma y, \\
-2\mathrm{B} &= \mathrm{A}\Sigma y + \Sigma y^2, \\
-3\mathrm{C} &= \mathrm{B}\Sigma y + \mathrm{A}\Sigma y^2 + \Sigma y^3, \\
-4\mathrm{D} &= \mathrm{C}\Sigma y + \mathrm{B}\Sigma y^2 + \mathrm{A}\Sigma y^3 + \Sigma y^4, \\
-5\mathrm{E} &= \mathrm{D}\Sigma y + \mathrm{C}\Sigma y^2 + \mathrm{B}\Sigma y^3 + \mathrm{A}\Sigma y^4 + \Sigma y^5, \\
&\quad \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.}
\end{aligned}
$$

And if in these equations we write $l, m, p, q, r,$ &c. in place of A, B, C, D, E, &c. respectively, and x in place of y, we obtain another set of equations, from which, by successive deductions, the sums of the powers of x may be obtained in terms of the coefficients of the equation in x.

Or these sums may be calculated, with perhaps greater facility, by the following method.

The first derived function of

$$x^n + lx^{n-1} + mx^{n-2} + px^{n-3} + \text{\&c.}$$

is $nx^{n-1} + (n-1)\,lx^{n-2} + (n-2)mx^{n-3} + \text{\&c.}$

and the quotient of the former function by the latter is, by a known theorem, equal to

$$\Sigma\,\frac{1}{x - x_1}.$$

Whence, by actual division, and equating the same powers of x, we obtain

$$
\begin{aligned}
\Sigma x &= -l, \\
\Sigma x^2 &= l^2 - 2m, \\
\Sigma x^3 &= -l^3 + 3lm - 3p, \\
\Sigma x^4 &= l^4 - 4l^2m + 2m^2 + 4lp - 4q, \\
\Sigma x^5 &= -l^5 + 5l^3m - 5lm^2 - 5l^2p + 5mp + 5lq - 5r, \text{ \&c.}
\end{aligned}
$$

Some very valuable tables of the symmetric functions up to the tenth degree of the roots of an equation of any order are given by Meyer Hirsch in his Algebra. Mr. Cayley has corrected Hirsch's Tables, and joined to them others, also of great value, giving reciprocally the expressions of the powers and products of the coefficients in terms of the symmetric functions of the roots. See a Memoir on the Symmetric Functions of the Roots of an Equation in the *Philosophical Transactions* for 1857.

The foregoing formulæ are perfectly general, and enable us to solve the comprehensive problem enunciated at the outset. Let us now consider the particular case proposed, in which

$$l = 0, \quad m = 0, \quad n = 5.$$

Here, observing that for all values of h not less than 5,

$$-\Sigma x^h = p\Sigma x^{h-3} + q\Sigma x^{h-4} + r\Sigma x^{h-5},$$

we find

$$
\begin{aligned}
\Sigma x &= 0, & \Sigma x^7 &= 7pq, \\
\Sigma x^2 &= 0, & \Sigma x^8 &= 8pr + 4q^2, \\
\Sigma x^3 &= -3p, & \Sigma x^9 &= -3p^3 + 9qr, \\
\Sigma x^4 &= -4q, & \Sigma x^{10} &= -10p^2q + 5r^2, \\
\Sigma x^5 &= -5r, & \Sigma x^{11} &= -11p^2r - 11pq^2, \\
\Sigma x^6 &= 3p^2, & \Sigma x^{12} &= 3p^4 - 24pqr - 4q^3,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma x^{13} &= 13p^3q - 13pr^2 - 13q^2r, \\
\Sigma x^{14} &= 14p^3r + 21p^2q^2 - 14qr^2, \\
\Sigma x^{15} &= -3p^5 + 45p^2qr + 15pq^3 - 5r^3, \\
\Sigma x^{16} &= -16p^4q + 24p^2r^2 + 48pq^2r + 4q^4, \\
\Sigma x^{17} &= -17p^4r - 34p^3q^2 + 51pqr^2 + 17q^3r, \\
\Sigma x^{18} &= 3p^6 - 72p^3qr - 36p^2q^3 + 18pr^3 + 27q^2r^2, \\
\Sigma x^{19} &= 19p^5q - 38p^3r^2 - 114p^2q^2r - 19pq^4 + 19qr^3, \\
\Sigma x^{20} &= 20p^5r + 50p^4q^2 - 120p^2qr^2 - 80pq^3r - 4q^5 + 5r^4.
\end{aligned}
$$

And by substitution, the formulæ for Σy, Σy^2, Σy^3, give

$$
\begin{aligned}
\Sigma y &= 5a - 3pd - 4q, \\
\Sigma y^2 &= 5a^2 - 6pad - 8qa - 6pbc - 8qbd - 10rb - 4qc^2 \\
&\qquad - 10rcd + 6p^2c + 3p^2d^2 + 14pqd + 8pr + 4q^2, \\
\Sigma y^3 &= 5a^3 - 9pa^2d - 12qa^2 - 18pabc - 24qabd \\
&\qquad - 30rab - 12qac^2 - 30racd + 18p^2dc + 9p^2ad^2 \\
&\qquad + 42pqad + 24pra + 12q^2a - 3pb^3 - 4qb^2c \\
&\qquad - 15rb^2d + 9p^2b^2 - 15rbc^2 + 18p^2bcd + 42pqbc \\
&\qquad + 21pqbd^2 + 48prbd + 24q^2bd - 9p^3b + 27qrb \\
&\qquad + 3p^2c^3 + 21pqc^2d + 24prc^2 + 12q^2c^2 + 24prcd^2 \\
&\qquad + 12q^2cd^2 - 18p^3cd + 54qrcd - 30p^2qc + 15r^2c \\
&\qquad - 3p^3d^3 + 9qrd^3 - 30p^2qd^2 + 15r^2d^2 - 33p^2rd \\
&\qquad - 33pq^2d + 3p^4 - 24pqr - 4q^3.
\end{aligned}
$$

Whence, by successive substitutions in the formulæ for A, B, C, we have

$$A = -5a + 3pd + 4q,$$

$$B = 10a^2 - 12pad - 16qa + 3pbc + 4qbd + 5rb + 2qc^2 \\ + 5rcd - 3p^2c + 3p^2d^2 + 5pqd - 4pr + 6q^2,$$

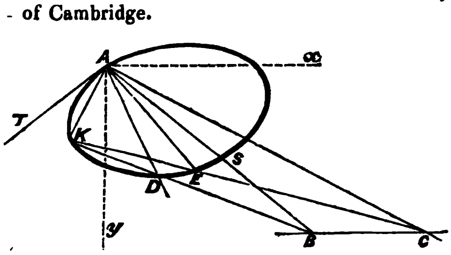
$$C = -10a^3 + 18pa^2d + 24qa^2 - 9pabc - 12qabd \\ - 15rab - 6qac^2 - 15racd + 9p^2ac - 9p^2ad^2 \\ - 15pqad + 12pra - 18q^2a + pb^3 + 4qb^2c \\ + 5rb^2d - 3p^2b^2 + 5rbc^2 + 3p^2bcd - 2pqbc \\ + 5pqbd^2 - prbd + 8q^2bd + 3p^3b + 11qrb \\ - p^2c^3 - pqc^2d - 8prc^3 + 4q^2c^2 + 7prcd^2 - 4q^2cd^2 \\ - 3p^3cd + 2qrcd - 2p^2qc - 5r^2c + p^3d^3 - 3qrd^3 \\ + p^2qd^2 - 5r^2d^2 - p^2rd + pq^2d - p^4 - 8pqr + 4q^3.$$

In the same manner, the values of Σy^4 and Σy^5, and thence also those of D and E may be calculated. The values of A, B, and C, exhibited above, coincide with those given by Mr. Bills in his able solution of Question 1381. (see p. 8.) The method here employed does not essentially differ from that by which Mr. Bills arrived at his results; he limited his attention however to the particular case proposed.

[NOTE.—Mr. Harley has pointed out how, by the aid of the foregoing investigation, Mr. Bills's ingenious simplification of Bring's process may be extended so as to effect the solution of the general problem which we have proposed as Question 1461.—EDITOR.]

1409 (Proposed by Mr. W. K. CLIFFORD.)—For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point K on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary.

Solution by A. CAYLEY, F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge.



I find that this very elegant theorem depends on the lemma to be presently stated, and that it is intimately connected with Newton's theorem for the organic description of a conic, or, what is the same thing, with the theorem of the anharmonic relation of the points of a conic.

LEMMA. If AT be the tangent, and AS any other line through a point A of a conic; and if two lines equally inclined to AT and AS respectively meet the conic in the points K and D (viz., if $\angle$ TAK = SAD, the two angles being measured in opposite directions from AT, AS respectively), then the line KD meets AS in a fixed point B, that is, a point the position of which is independent of the magnitude of the equal angles.

To prove this, take A for the origin, and the bisectors of the angle TAS for the axes of x and y: then the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2fy + 2gx = 0;$$

the equation of the tangent at the origin, that is, the line AT, is

$$gx + fy = 0;$$

and hence the equation of the line AS is

$$gx - fy = 0.$$

Taking $y = \alpha x$ for the equation of the line AK, we have for the coordinates x_1, y_1 of the point K where this meets the conic

$$(a + 2h\alpha + b\alpha^2)\, x_1 + 2\,(f\alpha + g) = 0, \quad y_1 = \alpha x_1;$$

and then the equation of the line AD will be $y = -\alpha x$, and we shall have for the coordinates x_2, y_2 of the point D where this meets the conic

$$(a - 2h\alpha + b\alpha^2)\, x_2 + 2\,(f\alpha + g) = 0, \quad y_2 = -\alpha x_2.$$

The equation of the line KD is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & \alpha x_1, & 1 \\ x_2, & -\alpha x_2, & 1 \end{vmatrix} = 0,$$

that is,

$$\alpha x\,(x_1 + x_2) + y\,(x_2 - x_1) - 2\alpha x_1 x_2 = 0;$$

and for the coordinates of the point B where this meets the line AS, the equation whereof is $gx - fy = 0$, we have

$$x\left\{f\alpha\,(x_1 + x_2) + g\,(x_2 - x_1)\right\} - 2f\alpha x_1 x_2 = 0,$$

or, as this may be written,

$$x\left\{\frac{f\alpha + g}{x_1} - \frac{-f\alpha + g}{x_2}\right\} - 2f\alpha = 0.$$

But we have

$$\frac{f\alpha + g}{x_1} = -\tfrac{1}{2}\,(a + 2h\alpha + b\alpha^2),$$

$$\frac{-f\alpha + g}{x_2} = -\tfrac{1}{2}\,(a - 2h\alpha + b\alpha^2);$$

and hence the equation is

$$x\,(-2h\alpha) - 2f\alpha = 0,$$

giving $x = -\dfrac{f}{h}$, and thence $y = -\dfrac{g}{h}$, for the coordinates of the point B; and these being independent of α, the lemma is seen to be true.

Consider now the points A, K as fixed points on the conic, and revolving about A the constant angle DAB, and about K the constant (zero) angle DKB; the locus of B is (by the theorem of the anharmonic relation of the points of a conic) given in the first instance as a conic through the points A, K; but observing that a position of the angle DAB is TAK, and that the corresponding position of DKB is AKA, the line AK is part of the

locus; and the locus is made up of this line and a line BC. And conversely, given the fixed points A, K, and the line BC, the original conic is, by Newton's theorem, described by means of the constant angles DAB, DKB revolving about these points in such a manner that the arms AB, KB generate by their intersections the line BC. This being so, the other two arms AD, KD generate by their intersections the conic.

And then, considering the two positions DAB, EAC of the angle DAB (so that D, B are in a line with K, and E, C are also in a line with K), we have $\angle DAB = EAC$, that is $\angle DAE = BAC$, which is Mr. Clifford's theorem.

It has been seen that, A being given, the same line BC is obtained whatever be the position of the point K; and taking AK for the normal at A, it at once appears geometrically that (as remarked by Mr. Clifford) the line BC is the polar of the point Θ of intersection of all the chords which subtend a right angle at A.

[NOTE.—Mr. Cayley's lemma may be otherwise proved as follows:—

The trilinear equation of the conic, referred to two tangents (α at A, β at S) and their chord of contact (γ or AS), is

$$U = \lambda\alpha\beta - \gamma^2 = 0;$$

and the equation of two straight lines (AK, AD) equally inclined to α, γ is

$$(\alpha - \mu\gamma)(\mu\alpha - \gamma) = 0, \text{ or}$$

$$V = \alpha^2 + \gamma^2 - (\mu + \mu^{-1})\,\alpha\gamma = 0;$$

also $U + V = 0$ denotes a conic passing through the intersections of U and V; but $U + V$ is resolvable into $\alpha = 0$, or the tangent AT, and

$$\alpha + \lambda\beta - (\mu + \mu^{-1})\,\gamma = 0,$$

which is therefore the equation of the chord KD; whence we see that KD meets AS (or γ) in a point B (given by $\gamma = 0$, $\alpha + \lambda\beta = 0$) whose position is independent of μ, that is, of the equal angles SAD, TAK.—EDITOR.]

1441 (Proposed by Dr. SALMON, F.R.S., Trinity College, Dublin.)—A pair of dice is thrown, or a teetotum of n sides is spun, an indefinite number of times, and the numbers turning up are added together (as in the game of Steeplechase); what is the chance (or rather the limit of the chance) that a given high number will be actually arrived at? For instance, if the game was won by whoever first got 100, and that getting 101 or 102 would not do.

Solution by the PROPOSER.

Take a single die of six sides (the method is the same if we have n instead of *six*), and let x be the chance of ever arriving at 100; then x will also be the chance of arriving at 101, 102, &c., since, when the number is large, it is evident that the chance of arriving at one high number is the same as that of arriving at another. Now the *first* number with three figures must be either 100, 101, 102, 103, 104, or 105. The chance that the first number with three figures should be 100 is x; since, if we arrive at 100 at all, it must be the first. The chance that 101 is the first is $\frac{5}{6}x$; for we might have come to 101 either from 100, or from one of the five preceding numbers. Hence we must have

$$x + \tfrac{5}{6}x + \tfrac{4}{6}x + \tfrac{3}{6}x + \tfrac{2}{6}x + \tfrac{1}{6}x = 1;$$

$$\therefore\ x = \tfrac{2}{7} = \text{the required chance.}$$

It might be shorter, though perhaps not so logical, to say that the *average* throw with a single die being $3\frac{1}{2}$, the chance is therefore

$$\frac{1}{3\frac{1}{2}} \text{ or } \frac{2}{7}.$$

1442 (Proposed by Dr. HIRST, F.R.S.)—The same circle around the origin being employed in the operations of reciprocation and inversion, show that the first positive and negative pedals of a given curve coincide, respectively, with the inverse of its reciprocal, and with the reciprocal of its inverse; further, that the reciprocal of the nth pedal is the $(-n)$th pedal of the reciprocal, and the $(-n-1)$th pedal of the inverse of the primitive; and lastly, that the inverse of the nth pedal is the $(-n)$th pedal of the inverse, and hence also the $(-n+1)$th pedal of the reciprocal of the primitive.

Solutions (1) by the PROPOSER; (2) by Mr. W. K. CLIFFORD, Trinity College, Cambridge.

1. The first part of this theorem is an immediate consequence of the definitions of inverse, reciprocal, and pedal curves. In fact, if m_1 be the foot of the perpendicular on the tangent at a point m of the primitive curve, the point m', inverse to m_1, with respect to a fixed circle around the origin, is the pole, with respect to that circle, of the line mm_1; that is to say, the point m_1, on the first positive pedal, is the inverse of a point m' on the reciprocal of the primitive curve. On the other hand, regarding m_1 as a point on the primitive, the envelope of m_1m, which is the first negative pedal, is at once seen to be the reciprocal of a point m' on the inverse of the primitive locus of m_1.

If we now take the reciprocal of each curve of a complete series of pedals

$$C_{-n},\ C_{-n+1},\ \ldots\ C_{-1},\ C,\ C_1,\ \ldots\ C_{n-1},\ C_n,$$

we shall obtain another series of pedals

$$C'_n,\ C'_{n-1},\ \ldots\ C'_1,\ C',\ C'_{-1},\ \ldots\ C'_{-n+1},\ C'_{-n},$$

arranged in a contrary order. For C_1, being the first positive pedal of C, is, according to the above, the inverse of C', the reciprocal of C. Hence the reciprocal of C_1 will be the reciprocal of the inverse of C', that is to say, the first negative pedal of C', and must accordingly be denoted by C'_{-1}. Similarly the reciprocal of C_2 is C'_{-2}, the first negative pedal of C'_{-1}, and so on.

A mere inspection of the above two series—in

the second of which each term is the reciprocal of the one above it in the first, and the inverse of the one above its right-hand neighbour,—is sufficient to establish the theorems in the question, which theorems are true for all positive or negative integral values of n. To take one example, it is seen that the inverse of the nth pedal C_n is C'_{-n+1}; that is to say, it is the $(-n)$th pedal of the inverse C'_1 of the primitive curve C; or what is equivalent, the $(-n+1)$th pedal of the reciprocal C' of the primitive.

2. Writing J for the operation of inversion, R for that of reciprocation, and P for that of taking the pedal, we have, by definitions and the first part of art. 1,

$$\begin{aligned} J^2 &= R^2 = 1, \\ P &= JR; \\ JP &= J^2R = R, \\ RJP &= R^2 = 1, \\ RJ &= P^{-1}. \end{aligned}$$

3. These, then, are the laws of combination of the symbols R, J, P. We can now immediately prove the theorems in the question. For

$$\begin{aligned} R\,.\,P^n &= R\,(JR)^n = (RJ)^n.\,R = P^{-n}R \\ &= P^{-n}.\,RJ\,.\,J = P^{-n-1}.\,J \;\ldots\ldots\ldots (1), \end{aligned}$$

$$\begin{aligned} J\,.\,P^n &= J\,(JR)^n = J^2R\,.\,(JR)^{n-1} = R\,(JR)^{n-1}J\,.\,J \\ &= (RJ)^n.\,J = P^{-n}J = P^{-n}.\,JR\,.\,R = P^{-n+1}.\,R \;(2). \end{aligned}$$

And we may write down any number of formulæ by this method. For instance, the identities

$$(JR)^n.\,J\,(RJ)^m = (JR)^{n+m}.\,J = J\,(RJ)^{n+m} \ldots (3),$$

$$(RJ)^n.\,R(JR)^m = (RJ)^{n+m}.\,R = R\,(JR)^{n+m} \;..\,(4),$$

may be thus interpreted :—

"The $(n)^{\text{th}}$ pedal of the inverse of the $(-m)^{\text{th}}$ pedal is the $(n+m)^{\text{th}}$ pedal of the inverse, and the inverse of the $(-n-m)^{\text{th}}$ pedal; and the $(-n)^{\text{th}}$ pedal of the reciprocal of the $(m)^{\text{th}}$ pedal is the $(-n-m)^{\text{th}}$ pedal of the reciprocal, and the reciprocal of the $(n+m)^{\text{th}}$ pedal."

4. Again; any formula may be transformed by interchanging R and J, and reversing the signs of all indices of P. To derive in this way the second pair of theorems from the first, we shall have to make a further change from n to $-n$. The formulæ (3) and (4) are immediately convertible.

5. The theory of Derived Surfaces and Curves is simply that of the interpretation of symbols. Let any straight line meet two rectangular axes Ox, Oy in A, B, and draw OP perpendicular to AB, and PM, PN perpendicular to the axes. Then we have two systems of coordinates; (1) when $\frac{1}{OA}$, $\frac{1}{OB}$ are the coordinates of the point P, (2) when PM, PN are the coordinates of the line AB. The formulæ of transformation, between the first and Cartesian, and between the second and Tangential, coordinates, are

$$(\xi^2+\eta^2)\,(x^2+y^2) = 1, \quad \xi y = \eta x \ldots\ldots\ldots (5).$$

These represent the operation of inversion in the two cases. But it is important to remember that, in *Tangential inversion*, the tangents, not the points, are inverted; that is, to every tangent of the primitive corresponds a line parallel to it, such that the rectangle under their distances from the origin is constant. Now let $U=0$ be an equation in x, y, and constants; and let CU denote the curve which is represented by $U=0$, when we interpret x, y as *Cartesian* coordinates, TU when we interpret x, y as *Tangential* coordinates, MU according to the *first* system of this article, and NU according to the *second*. Then, for instance, TU = R . CU, or, by separation of symbols, T = RC. In this way we have the equations

$$M = JC, \quad N = RJC = RM = RJRT,$$

which serve to connect any two systems.

6. It appears from (5) that if the equation of any curve be written

$$u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0 = 0,$$

then the equation of its inverse is

$$\begin{aligned} u_n + u_{n-1}(x^2+y^2) + u_{n-2}(x^2+y^2)^2 + \ldots \\ + u_0\,(x^2+y^2)^n = 0. \end{aligned}$$

This is of degree $2n$ in general, but reduces when the curve is circular, and when the origin is on the curve. If the curve be circular in the degree f, that is, if its equation be of the form

$$\begin{aligned} v_{n-2f}(x^2+y^2)^f + v_{n-2f+1}(x^2+y^2)^{f-1} + \ldots \\ + u_1 + u_0 = 0, \end{aligned}$$

the degree of the inverse is reduced $2f$. And if the origin be a multiple point of the order g, or if

$$u_0 = u_1 = \ldots = u_{g-1} = 0,$$

the degree is reduced by g. Hence generally the degree of the inverse is $2(n-f)-g$. It follows by reciprocation that if n is the class of any curve, and if the lines joining the origin to the circular points at infinity are multiple tangents of the order f, and if the line at infinity is a multiple tangent of the order g, then the degree of the first positive pedal is $2(n-f)-g$. And again; if a curve has g points at infinity distinct from the two circular points at infinity, and has a multiple point of the order ϕ at the origin, being of degree v, then the *class* of the first negative pedal is $v-\phi+g$. This is easily obtained by inverting the result just proved; it being remarked that the inverse of a curve circular in the degree f has a multiple point of the order $n-2f$ at the origin. The *degree* of the negative pedal is the *class* of the inverse, and consequently is the same as the number of circles which can be drawn through an arbitrary point (ξ, η) and the origin to touch the primitive curve. To find this number, we must eliminate between

$$U = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x^2+y^2+2Ax+2By = 0 \ldots\ldots\ldots\ldots (2),$$

$$(x+A)\,\frac{dU}{dx} = (y+B)\,\frac{dU}{dy} \ldots\ldots\ldots\ldots (3),$$

A and B being connected by the linear relation

$$\xi^2+\eta^2+2A\xi+2B\eta = 0 \quad \text{..........}(4).$$

The degree of the eliminant in A and B, which is the degree of the first negative pedal, is *in general* $n(n+2)$, but will of course be reduced by peculiarities in the form of U.

1445 (Proposed by CANTAB.)—Integrate, by Charpit's method, the equation

$$(mz-ny)\,p + (nx-lz)\,q = ly-mx.$$

(See Boole's Differential Equations, p. 342.)

Solution by J. McDOWELL, B.A., F.R.A.S.

The type-form is

$$\frac{dx}{-\frac{dq}{dp}} = dy = \frac{dz}{q-p\frac{dq}{dp}} = \frac{dp}{\frac{dq}{dx}+p\frac{dq}{dz}}.$$

By substitution this becomes

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$= \frac{(nx-lz)\,dp}{(l+np)\left\{(mz-ny)+p\,(ly-mx)\right\}};$$

$$\therefore\ (l+np)\,(dx+pdz) + (lz-nx)\,dp = 0,$$

which may be put into the form

$$\frac{(ndx-ldz)+(l+np)\,dz}{nx-lz} = \frac{ndp}{l+np};$$

and, putting $nx-lz = u$, $l+np = v$, it becomes

$$\frac{du+vdz}{u} = \frac{dv}{v}, \text{ or } dz+\frac{vdu-udv}{v^2} = 0;$$

$$\therefore\ z+\frac{u}{v} = z+\frac{nx-lz}{l+np} = const.\ (=cn \text{ suppose});$$

whence we have $p = -\dfrac{x-cl}{z-cn}$,

and $q = \dfrac{(ly-mx)-p\,(mz-ny)}{nx-lz} = -\dfrac{y-cm}{z-cn}$.

Hence $dz = pdx+qdy$ gives

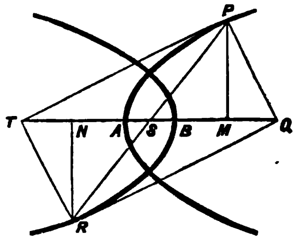
$$(x-cl)\,dx+(y-cm)\,dy+(z-cn)\,dz = 0;$$

therefore, integrating, we find

$$(x-cl)^2+(y-cm)^2+(z-cn)^2 = r^2.$$

1411 (Proposed by R. TUCKER, M.A.)—The envelop of a perpendicular drawn to a normal of the point P of a parabola, from the point where it cuts the axis, is a parabola. Show that the focal vector of P meets the envelop where the perpendicular touches it.

Solution by Mr. JAMES WILSON; Mr. W. K. CLIFFORD; and R. TUCKER, M.A.



Let PT be a tangent and PQ a normal to a parabola (AP) whose vertex is A and focus S; draw QR perpendicular to PQ, or parallel to PT, meeting PS produced in R; join RT, make SB = SA, and draw PM, RN perpendicular to TAQ.

Then ST = SP = SQ = SR, PQRT is a rectangle, TN = QM, and NB = MA. Hence QR obviously *touches* at R a parabola (BR) equal to AP, but turned the other way, having B for its vertex, and S for its focus.

1294 (Proposed by MATTHEW COLLINS, B.A., Senior Moderator in Mathematics and Physics, Trinity College, Dublin.)—If five circles A, B, C, D, E, pass through one point P, and if five other circles be described about the five circular triangles formed by ABC, BCD, CDE, DEA, and EAB (P not being a corner of any of the five circular triangles), these five new circles will intersect each other consecutively in five new points, not lying on the five original circles A, B, C, D, E; prove that these five new points lie all upon the circumference of one circle.

1297 (Proposed by MATTHEW COLLINS, B.A.) If four circles A, B, C, D, pass through one point P, and if (AB) denote the *other* point of intersection of the circles A and B; then describe a circle through P, (AB), (CD), cutting a new circle orthogonally at (AB) and (CD); describe also a circle through P, (AC), (BD), cutting another new circle orthogonally at (AC), (BD); and describe a third circle through P, (AD), (BC), cutting a third new circle orthogonally at (AD), (BC); then prove that these three new circles have a common radical axis.

1306 (Proposed by MATTHEW COLLINS, B.A.) When four straight lines intersect, they form four triangles, and the circles described about these four triangles pass through one point. Conversely, when four circles, A, B, C, D, pass through one point P, the four circular triangles formed by them have each three corners (P being none of these corners); and if the three corners of each of any two of these triangles lie *in directum*, prove that the same must be true for each of the two remaining circular triangles. Prove also, in the general case, that the four circles described about these four circular triangles must all pass through one point P'.

1316 (Proposed by MATTHEW COLLINS, B.A.) If four circles A, B, C, D pass through one point P, every three of them form a circular triangle.

Let D′ be the point of intersection of three circles passing through P, and through the corners of the circular triangle formed by ABC, cutting the opposite sides orthogonally; and let C′, B′, A′ be the analogous points for the other three circular triangles, formed without C, B, A respectively. Prove that the five points A′, B′, C′, D′, P lie upon one circle (Q). Moreover if (AB) denote the other point (not P) of intersection of the circles A, B, and if P_1 be the harmonic conjugate point of P on the circle passing through the points P, (AB), (CD), relative to the two latter points, P_2 the analogous point on the circle through P, (AC), (BD), and P_3 the analogous point on the circle through P, (AD), (BC); then prove also that the four points P, P_1, P_2, P_3 lie all upon the circumference of another circle (R), which cuts (Q) orthogonally.

Solution by W. J. Miller, B.A., Mathematical Master, Huddersfield College.

These theorems are deducible from known theorems by the method of *inversion*, or *transformation by reciprocal radii vectores*, which consists in taking, on every radius vector (PI) drawn from any origin to a point (I) in a curve or system of lines, a part (PJ) equal to its *reciprocal* or *inverse*, that is, such that the rectangle PI . PJ may be equal to an assumed square (unit).

In this method of transformation, the *inverse* of a straight line is, *in general*, a circle passing through the origin, having its centre in the perpendicular from the origin on the line, and *vice versâ*; the inverse of any circle not passing through the origin is a circle, the origin being one of the centres of *similitude* or *perspective* of the two circles; a straight line through the origin has itself for inverse; the angle contained by two intersecting lines (curved or straight) is equal to that contained by their inverses; and the distance between two points is to the distance between their inverses as the assumed square unit is to the rectangle contained by the radii vectores of these inverse points. From this last property it can be easily shown that the inverse of the *middle* (L) of a *finite* straight line (AC) is the harmonic conjugate point (L′) of the origin (P), on the circle (PA′L′C′) which is the inverse of the given straight line, relative to the points (A′, C′) which are the inverses of the ends of the line.

The method of inversion, applied to surfaces, includes *Stereographic Projection* as a particular case; and properties analogous to the foregoing, with many others, subsist for *inverse surfaces*. An excellent summary of these properties is given in the *Note* on pp. 35, 36 of Dr. Hirst's Memoir on *Pedal Curves and Surfaces*, in *Tortolini's Annali* for 1859.

The theorems in the above Questions may be derived from the following, which are proved in Mr. Wilkinson's Solutions of Questions 1965, 1878, 1847 of the "*Lady's and Gentleman's Diary*."

I. If the sides of a pentagonal figure be produced to meet, they will form five triangles whose bases are the sides of the original figure; and if circles be described about the triangles thus formed, they will intersect, two and two, in five points, which lie upon the same circle.

II. The circles described upon the three diagonals of a complete quadrilateral, as diameters, have the same radical axis.

Moreover, the four points of intersection of the altitudes of the four component triangles of the complete quadrilateral all lie on the aforesaid radical axis, and are therefore in a straight line perpendicular to that which joins the middles of the diagonals.

In the language of the *Modern Geometry*, these theorems may be enunciated as follows:

The three circles of which the three chords of intersection of any four lines are diameters are coaxal, and therefore the middle points of the chords are collinear; also the four polar centres of the four triangles determined by the four lines are on the radical axis of the aforesaid three circles, and are therefore collinear; and further, the two lines of collinearity for the middle points of the chords and for the polar centres of the triangles are orthogonal. Under this form the theorems are proved in *Townsend's Modern Geometry*, vol. i., art. 189.

III. The four circles described about the four component triangles of a complete quadrilateral pass through the same point.

Now, by inverting (I), we obtain 1294. This is easily seen, and needs no remark.

Inverting the first part of (II) with respect to an origin in the radical axis, but not on the circles, we have the theorem in Question 1297; since the radical axis remains unchanged by the transformation, and is evidently the radical axis of the three "*new circles*," which are the inverses of those on the diagonals of the original quadrilateral as diameters, and which cut orthogonally the three circles through P, the inverses of the three diagonals themselves.

The inverse of the *second* part of (II) is the theorem in Question 1316: for A′, B′, C′, D′ are the inverses of the four points of intersection of the altitudes (that is, the *polar centres*) of the four component triangles of the original quadrilateral; hence these four points and the origin (P) lie upon one circle (Q), which is the inverse of the radical axis of the three circles on the diagonals as diameters; also P_1, P_2, P_3 are the inverses of the *centres* of these circles, and therefore the four points P, P_1, P_2, P_3 lie upon the circumference of a circle (R) which cuts (Q) orthogonally. (See Mr. Collins's Solution of Question 982 in the *Educational Times* for June, 1863.)

Inverting (III) with respect to an origin not in the circumference of any one of the four circles, we obtain the last part (or *general case*) of 1306; and, in the *first* part, it is obvious that the origin must be at the point of intersection of *two* of the circles, and therefore of the *whole four*; whence the first part follows, and the theorem is proved.

1319 (Proposed by N'IMPORTE.)—It is announced at p. 205, vol. ii., 12th edition, Davies's Hutton, that "if a tetrahedron be drawn, formed of four tangent planes to a paraboloid, the sphere described about it will pass through the focus of the paraboloid." Prove or disprove this.

Solution by Mr. W. K. CLIFFORD, Trinity College, Cambridge.

The statement is not true.

If perpendiculars be drawn from the foci of a conicoid of revolution on any tangent plane, the rectangle of these perpendiculars is equal to the square of the minor axis. If then a conicoid of revolution, having foci $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$, touch the faces of the fundamental tetrahedron, we must have $\alpha\alpha_1=\beta\beta_1=\gamma\gamma_1=\delta\delta_1=b^2$. So that if one of the foci lies on the plane

$$l\alpha_1+m\beta_1+n\gamma_1+r\delta_1=0,$$

the locus of the other will be the surface of the third degree

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}+\frac{r}{\delta}=0 \quad \ldots\ldots\ldots\ldots(1),$$

which is otherwise interesting. (*Frost and Wolstenholme's Solid Geometry*, p. 289.) A particular case is when the surface of revolution is a paraboloid, one of whose foci lies on the plane at infinity, $A\alpha+B\beta+C\gamma+D\delta=0$, and the other on the surface

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}+\frac{D}{\delta}=0 \quad \ldots\ldots\ldots\ldots(2),$$

where ABCD are the faces of the tetrahedron. Now if the theorem of Davies's Hutton were true, we should have found for the locus the equation of the circumscribing sphere.

I write down one or two other instances of the application of this principle. (See *Salmon's "Conics,"* 4th Ed. p. 261. Exs. 13, 15.)

Given five planes connected by the identical relation

$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0,$$

the foci of any conicoid of revolution touching the "frustum" will lie in the surface of the fourth degree,

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}+\frac{e}{\epsilon}=0.$$

Given one focus and the intersection of the focal tangents of a parabola of the third class inscribed in the triangle of reference; the other focus moves on a conic (which is *never* a circle) circumscribing the triangle.

Given four tangents, a focus, and the intersection of the focal tangents, in a curve of the third class; the other two foci move on a curve of the third degree.

The general extension is sufficiently obvious.

1421 (Proposed by Professor SYLVESTER, F.R.S., Royal Military Academy, Woolwich.)—If by the harmonic centre, relative to a fixed plane, of A, C, points in a line meeting the fixed plane in D, be understood a point B between A and C, such that A, B, C, D form an harmonic system; prove that if through the harmonic centre of either diagonal of any of the three quadrilateral faces of the frustum of a triangular pyramid, and the harmonic centres of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will all pass through one and the same point.

Solution by Mr. W. K. CLIFFORD, Trinity College, Cambridge.

The proposer has shown, in the "Philosophical Magazine" for September, that the theorem is true when we put the arithmetical centre for the harmonic. His proof, by Cartesian coordinates, is exceedingly simple and elegant; but it will be found that the attempt to prove the same case by Quadriplanar coordinates involves an enormous amount of algebraic work. However, it is clear that if the requisite operations were performed, the result must be the same as by the other method. Now when we find the arithmetic centre of a line by Quadriplanar coordinates, the process is simply to find its harmonic centre with respect to the plane at infinity $A\alpha+B\beta+C\gamma+D\delta=0$. But the proof can make no mention of the meaning of ABCD, since the thing proved is general for any tetrahedron, and does not depend at all upon the areas of the faces. *Therefore the proof holds, whatever interpretation we give to the symbols ABCD*; that is to say, whatever plane is represented by $A\alpha+B\beta+C\gamma+D\delta=0$.

This principle is evidently identical with the method of Projections in plane geometry. (See Salmon's Higher Plane Curves, Art. 246.) Quadriplanar equations not connected with the absolute form of the fundamental tetrahedron, will hold good whatever tetrahedron we choose. Now it is analytically possible to choose a tetrahedron with reference to which a given conicoid shall be represented by a given equation. For a conicoid is determined by nine conditions, but each of the four planes involves three independent constants. We may, therefore, in addition, choose the tetrahedron so that the plane at infinity shall be represented by a given equation. Thus any property proved of any one conicoid and a plane, when expressed in Quadriplanar coordinates, is true of any other concoid and plane. The only limitation is that connecting properties which can be expressed in the coordinates will be retained in transformation. Again, we may analytically transform real lines and planes into imaginary, and *vice versâ*, without loss of continuity. Now, ruled conicoids only differ from others in containing real lines instead of imaginary; therefore, *the distinction between ruled and unruled conicoids is lost in transformation.*

In studying, then, the properties of any figure, the principal point will be the reduction of the figure to what (by an extension of the term) may be called the *canonical form*. For instance, the canonical form of a quadrilateral is a parallelogram; of a conic, a circle; of a conicoid, a sphere; and so on. In particular, we wish to find the canonical form of a tetrahedral frustum, which is the figure

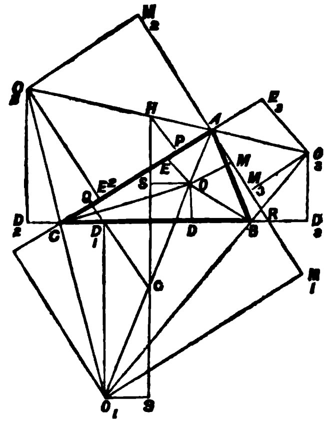
formed by the intersection of five planes ABCDE. For a quadrilateral ABCD we proceed in this way; we join the vertex $\widehat{AB}$ to the vertex $\widehat{CD}$, and project the joining line to infinity. Hence by analogy in the frustum, we join the vertex $\widehat{ABC}$ to the edge $\widehat{DE}$, and project the joining plane to infinity. The figure is thus reduced to three parallel straight lines cut by two parallel planes. As an instance of the use of these canonical forms, we give the following properties, which may be easily proved:—

"From a point O 3 chords o are drawn to a conicoid, meeting it in 6 points A. These may be joined again by 4 pairs of planes α, each pair intersecting in one of 4 lines β on the polar plane of O. At the points A 6 tangent planes b are drawn; if any three of these (whose points of contact are not in one plane through O) intersect in X and the other three in Y, then X, Y, O are in a straight line. Again, the planes b will intersect by pairs in 3 lines γ on the polar plane of O, and these will pass through the three intersections of one of the lines β with the other three. If the chords o cut the polar plane in 3 points, these will lie in 3 straight lines through the same intersections. There are thus 3 co-axal triangles on the polar plane, and their common pole is on the line joining O and two of the intersections XY. The tangent planes b cut the chords o in 12 new points C, four of which lie on each chord. Consider the eight C-points lying on two chords o: they may be divided into 2 groups, each group having two points on each of the chords. Lines joining points in *either* group intersect on the polar plane of O, but one group has for these two intersections (1) a vertex of one of the three co-polar triangles, (2) the point where the opposite side cuts the common axis. Points in different groups may be joined by 8 lines, intersecting in 4 points lying on the polar plane of O, and 8 points lying on 4 lines through O. And so on.

"If a straight line be drawn through the vertex of either of the common tangent cones of two conicoids having double contact, to meet either of the planes of common section, and the two conicoids, it will be cut in involution, so that the equal anharmonic ratios of the involution are constant."

1446 (Proposed by Mr. W. H. LEVY.)—Given the radius of *either* of the four circles of contact, the vertical angle, and the difference of the sides; to construct the triangle.

Solutions (1) by Mr. W. H. LEVY; ALPHA; and T. T. WILKINSON, F.R.A.S., Grammar School, Burnley; (2) by Mr. J. CONWILL, Leighlinbridge School; and W. O. PHILLIPS, M.A.



1. *First Solution.—Construction.* Make the angle BAC equal to the given vertical angle, and draw the bisectors AG, AH. In AC take AP equal to half the given difference of the sides; and through A and P draw AR and PH perpendicular to AC, the latter meeting AH in H. In AR take AM (or M_1, &c.) equal to the given radius, and draw MO (or O_1, &c.) parallel to AC, meeting AG or AH in O. If the point O be in the *internal* bisector (AG), with centre O and radius equal to AP, describe a circle, to which draw from H the tangent HS, meeting AG in G. With centre G or H, and radius GO, (or GO_1) or HO_2 (or HO_3), according as the point O is in the *internal* or *external* bisector, describe a circle cutting AB, AC in B and C: then ABC will be the triangle required.

Demonstration. Join OS, and draw the other lines as in the figure. Then by known properties (*See McDowell's "Exercises,"* Props. 73, 75, 77, 100) it may be easily shown that G, H are points in the circle circumscribing the triangle ABC, GH a diameter bisecting BC at right angles, O the centre of one of the circles of contact, and AP, OS, or CQ half the difference of the sides; moreover, by construction, BAC is equal to the given vertical angle, OD equal to the given radius, and AP equal to half the given difference of the sides; therefore ABC is the triangle required.

2. *Second Solution.—Analysis.* Let ABC be the required triangle, AG and AH the bisectors of the given vertical triangle (A), and O, O_1, O_2, O_3 the centres of the circles of contact. Then, if the *given* radius be that of one of the circles whose centres (O, O_1) are in the *internal* bisector (AG), we have given, in the triangle BOC (or BO_1C), (1) the vertical angle, which is greater (BOC) or less (BO_1C) than a right angle by half the given vertical angle (A), (2) the perpendicular OD (or O_1D_1), which is equal to the given radius, and (3) the difference of the segments BD, DC (or BD_1, D_1C_1) of the base, which is equal to the *given* difference of the sides AB, AC; hence this triangle is given (see *McDowell's "Exercises,"* Prop. 160) and therefore the required triangle ABC is determined.

If the *given* radius (O_2D_2) be that of a circle

whose centre (O_2) is in the *external* bisector (AH), the right-angled triangle AO_2E_2 is given, and therefore BC is given, since it is the difference between $2AE_2$ and the given difference (AP) of the sides AB, AC; hence, to construct the required triangle ABC, we have given the base, the vertical angle, and the difference of the sides, a well-known problem. (See *Catalan's Théorèmes et Problèmes*, p. 32, 3rd ed. Prob. xiv.; or *Townsend's Modern Geometry*, vol. i., art. 67.)

1454 (Proposed by Matthew Collins, B.A., Senior Moderator in Mathematics and Physics, Trinity College, Dublin.)—Through a given point P, within a given angle BAC, to draw a straight line BPC, so that the *Geometric, Harmonic,* or *Arithmetic* mean between the segments PB, PC may be given or a minimum.

Solutions by Archer Stanley; Alpha; Mr. W. Hopps; and Mr. J. Wilson.

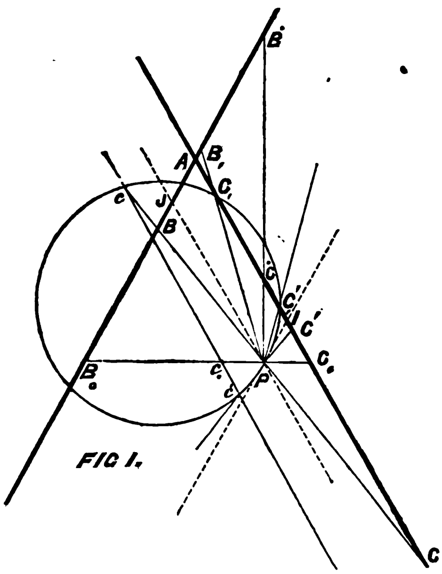


FIG 1.

1. Draw the circle Pcc' (Fig. 1) which is the *inverse* of one of the given lines AB with respect to a circle around P as centre, whose radius is the given geometric mean. The diameter through P of this circle is of course perpendicular to AB, and a third proportional to the distance of P from AB, and to the given mean. Each of the four lines PC_1, PC'_1, Pc, Pc', drawn to the points where this circle cuts the given line AC, and a parallel to the latter, equidistant from P, corresponds to a solution of the problem. For B_1, B'_1, B, B′ being the points in which these lines cut AB respectively, and C_1, C'_1, C, C′ those in which they cut AC, it follows from the definition of inverse curves that the square on the given mean is equal to each of the rectangles PB . PC (= PB . Pc), PB′ . PC′(=PB′ . Pc'), $PB_1 . PC_1$, and $PB'_1 . PC'_1$.

On two of the lines BC and B′C′ the segments lie on *opposite* sides of P, on the other two lines B_1C_1, $B'_1C'_1$ the segments lie on the *same* side. If we draw PI and PJ parallel to the given lines, it is evident that all lines of the *latter* class fall within the angle IPJ, and all lines of the *former* class within the angles adjacent to IPJ.

The series of circles inverse to AB, obtained by giving all possible values to the geometric mean, will include one which touches cc' in some point c_0, and another which touches AC in a point $\dot{C}$. These points c_0 and $\dot{C}$ will clearly determine lines B_0C_0, $\dot{B}\dot{C}$ which, in their respective classes, correspond to geometric means of *minimum* value. For as the given mean diminishes, so also does the diameter of the circle; and below the above limits no intersections with the lines cc' or AC are obtainable. The lines B_0C_0, $\dot{B}\dot{C}$ are at right angles to each other, and each is equally inclined to the given lines; for the condition of tangency above stated clearly requires IP, $I\dot{C}$, and IC_0 to be equal to one another; thus B_0PC_0, $\dot{B}\dot{C}P$ are the bisectors of the angle IPJ.

It is worth observing, lastly, that each pair of lines belonging to the same class are corresponding rays of an involution, and are equally inclined to the lines PC_0 and $P\dot{C}$, which are the double rays of that involution.

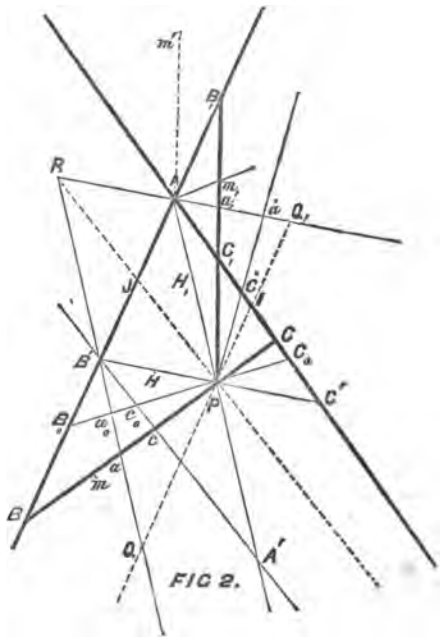


FIG 2.

2. The following construction of Fig. 2 applies to the second part of the question. Set off PA′ = PA, and through A′ and P draw parallels A′B′, PJ to one of the given lines AC. The line B′C′

will clearly be bisected in P. Through A draw a parallel to it, and through B′ a parallel to AP. These parallels will meet in a point R of PJ. It will be at once seen that each of the pencils $\{A.RBPC\}$, $\{B'.RBA'P\}$ is harmonic; so that any transversal BC of the *first* class (within the angle JPQ, and therefore divided *internally* into two segments by P) will be cut by RQ in a point a so that Pa will be the harmonic mean of PB and PC (= Pc); whilst every transversal B_1C_1 of the *second* class (within the angle IPJ, and consequently divided *externally* by P) will be cut by RQ_1 in a point a_1 so that Pa_1 will be the harmonic mean of the segments PB_1 and PC_1. If we conceive the transversal to rotate through two right angles, it will be evident that the harmonic means corresponding to transversals of the *first* class diminish from PQ to a *minimum* Pa_0 ($\perp$RQ), and then increase to a *maximum* PR, whilst the harmonic means for transversals of the *second* class diminish from this *maximum* PR to a *minimum* $P\dot{a}$ ($\perp RQ_1$,) and then increase to the limit PQ_1 (= PQ).

A circle around P as centre, with a radius equal to the *given* harmonic mean, will in general cut RQ in two points a, a', to which will correspond two transversals Pa, Pa' of the *first* class, and RQ_1 in two points a_1 and a'_1 corresponding to two transversals Pa_1, Pa' of the *second* class. The number of solutions, which cannot exceed four, depends of course upon the magnitude of the given mean, and may be easily ascertained in any particular case.

3. With respect to the third part of the question, the solution may be shown to depend upon the determination of the four intersections of a circle and a hyperbola passing through its centre. For Pm_1 being any variable transversal, let Am' be parallel to it, and let Am_1 be the harmonic conjugate of Am', with respect to the given lines AB_1, AC_1. Then, as the transversal rotates, the rays Am', Am_1 will clearly generate an involution of which AB_1 and AC_1 are the double rays. Consequently Am_1 and Pm_1 will generate two homographic pencils, and the locus of the intersection m_1 of corresponding rays will be a conic, in fact a hyperbola (H_1), with asymptotes parallel to the given lines. This hyperbola passes through A and P, and is touched at these points by the parallel lines AR and PB′, so that its centre H_1 bisects AP. All this is manifest on observing that AB_1 and PI, AC_1 and PJ, AP and PB′, AR and PA, are four pairs of corresponding rays of the two pencils. But m_1 bisects B_1C_1, so that for transversals of the second class the radii vectores Pm_1 of the hyperbola (H_1) are the arithmetic means of the corresponding segments PB_1, PC_1. Similarly the arithmetic means of the segments on transversals of the first class are the radii vectores of a hyperbola (H) passing through P and B′ (at which points PA and B′R are the tangents), having its centre at H, midway between P and B′, and its asymptotes also parallel to the given lines AB, AC.

A circle around P as centre, with a radius equal to the *given* arithmetic mean, wi l, in general, cut (H) as well as (H_1) in four points, the radii vectores to which will be transversals of the kind required. The number of real solutions, which cannot exceed eight, will however depend upon the magnitude of the mean. The normals let fall from P upon each hyperbola, of which there are in general three, will correspond to the maxima and minima of arithmetic means.

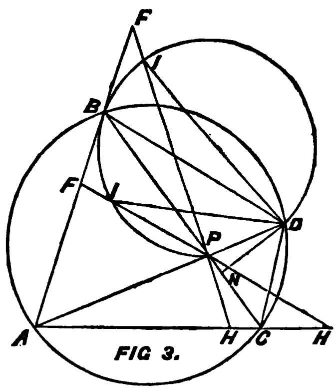


FIG 3.

4. Again, suppose BPC (Fig. 3) drawn as required for the *Geometric* mean, and through A, B, C describe a circle cutting the line passing through A, P in D, and join BD. Then AP . PD = BP . PC, which is given; but AP is given, therefore PD is given; also the angle PBD (= PAC) is given. Whence it is obvious that B is the intersection of AB with the circular arc on PD containing an angle equal to the given angle PAC: the position of BPC is therefore known. Now suppose the circular arc on PD to *touch* AB at B, and draw *any* straight line through P to meet AB, AC at F, H and cut the circular arc PBD in I, and draw ID. Then $\angle$PID = PBD = PAH, therefore A, I, D, H are in the circumference of a circle; consequently IP . PH = AP . PD = BP . PC; but IP . PH is less than FP . PH, therefore BP . PC is also less than FP . PH; that is, BP . PC is a *minimum*. Moreover, since $\angle$ABC = BDA (Euc. iii. 32) = BCA, it is obvious that B is the intersection of AB with the circular arc on AP containing an angle equal to half the supplement of the given angle BAC; also when BP . PC is *not* a minimum, the circular arc on PD cuts AB in two points, therefore in this case *two* positions of BPC may in general be found.

It is easy to apply the methods used in this and the next two Articles to the case in which the line BC is divided *externally* by P; this may, in fact, be reduced to the case of *internal* section by drawing parallels to the given lines, equidistant with them from P.

5. Next, suppose BPC drawn as required for the *Harmonic* mean; then

$$\tfrac{1}{2}BC : BP = PC : L,$$

L being a given line; therefore

$$BC . L = 2BP . PC = 2AP . PD;$$

hence BC : PD = 2AP : L = a given ratio . . (a). Draw DN perpendicular to BC, and join D, C; then the angle DBC (= DAC) is given, and the angle DCB (= DAB) is also given; therefore the triangle DBC is given in species; hence the ratio of BC to DN is given; but (a) the ratio of BC

to PD is given, therefore the ratio of DN to PD is given, and DNP is a right angle, consequently the angle DPN is given, which is the angle of inclination of AP, BC; hence the position of BPC is given. Now from (α) it is obvious that L is a *minimum* when the ratio of PD to BC is a minimum, which (as the triangle DBC is given in species) is manifestly the case when PD is perpendicular to BC, that is, when BPC is at right angles to AP.

6. Lastly, suppose BPC to be the line required for the *Arithmetic* mean; put $BC = b$, $AP = c$, $\angle BAP = \alpha$, $\angle PAC = \beta$, and $\angle BPA = \theta$: then

$$BP = \frac{c \sin \alpha}{\sin(\theta+\alpha)}, \text{ and } CP = \frac{c \sin \beta}{\sin(\theta-\beta)};$$

$$\therefore \frac{\sin \alpha}{\sin(\theta+\alpha)} + \frac{\sin \beta}{\sin(\theta-\beta)} = \frac{b}{c},$$

whence, by reduction, we obtain

$$r^2x^4 - 2rstx^3 + (t^2 - 2rsv - r^2u^2)x^2 + 2rtvx + r^2v^2 = 0,$$

where r, s, t, u, v represent the *known* quantities $(b : c)$, $\cos(\alpha-\beta)$, $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\sin\alpha \sin\beta$, and x represents the *unknown* quantity $\sin\theta$. Hence $\sin\theta$ may be found, which determines the position of BPC.

When $\alpha=\beta$, the equation becomes

$$rx^2 - (\sin 2\alpha)x - r\sin^2\alpha = 0;$$

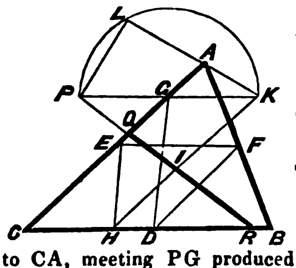
which applies to the case when P is in the bisector of the given angle BAC.

The following is a geometrical solution of this case, the diagram to which may be readily sketched. Assume the position of BPC to be known; and through A, B, C describe a circle cutting the bisector of the angle BAC in D; draw the diameter DF cutting BC in H, and join AF, DB, DC. Then $\angle CBD = CAD = BAD = BCD$; therefore the isosceles triangle DBC is given in species, and the side BC is given, hence BD is given. And as DF is a diameter, the right angled triangles DAF, DHP are similar, hence $DA \cdot DP = DF \cdot DH = DB^2$. We have, therefore, the *difference* and *rectangle* of the lines DA, DP given to determine them, which is a known problem. It is obvious that when BC is a minimum, BD and DP must each be a minimum; and BPC must then be perpendicular to AP.

Constructions of the general problem are also given by Mr. Miller and Mr. Bills in the *Educational Times* for March, 1863 (*Solution of Quest.* 1264); and another geometrical construction of the particular case when P is in the bisector of the angle BAC is given by Mr. Miller in the Solution of Question 1086 in the *Educational Times* for July, 1859; moreover, a very elegant construction and demonstration of the *minimum* position of BPC, for the general case, is given by Mr. Collins in the *Educational Times* for June, 1854 (*Solution of Quest.* 632).

1457 (Proposed by R. Palmer, M.A.)—To bisect a given triangle by a straight line drawn through a given point without it.

Solution by Dr. Rutherford, F.R.A.S., Royal Military Academy, Woolwich; and similarly by R. Palmer, M.A., W. O. Philips, M.A.; R. Tucker, M.A.; and Mr. J. Wilson.



Construction.—Let P be the given point, ABC the triangle, and D, E, F, the middles of the sides. Draw PGK parallel to CB, and join DG. Through E draw EH parallel to GD, and then draw HK parallel to CA, meeting PG produced in K. On PK describe a semicircle, and inflect in it PL equal to PG; join KL, make HR equal to KL, and draw PQR; then QR will bisect the triangle.

Demonstration.—Let QR and HK intersect in I; then the triangles PGQ, PKI, HIR are obviously similar; and since $PK^2 = PL^2 + LK^2$, and PL is equal to PG, and HR to LK, it is evident that the triangle PIK is equal to the two triangles PGQ, HIR; hence the quadrilateral QGKI is equal to the triangle HIR, and therefore the triangle CQR is equal to the parallelogram CK. But GD and EH are parallel, therefore

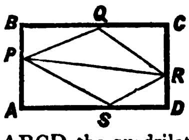
$$GC : CD = EC : CH:$$

hence the parallelograms CK and CF have a common angle C, and the sides about that angle reciprocally proportional; therefore the parallelograms are equal, and consequently the triangle CQR is equal to the parallelogram CF, which is half the triangle ABC.

Note.—About three years ago, I communicated the foregoing construction of this problem to my colleague Thomas Bradley, Esq., who published it in his excellent work, entitled *Elements of Geometrical Drawing*, which appeared in 1861.

1282 (Proposed by Mr. Stephen Watson.)—Four points are taken at random, one on each side of a square; what is the chance that the quadrilateral formed by joining them will be less than *three-fourths* of the square?

Solution by W. J. Miller, B.A., Mathematical Master, Huddersfield College.



1. Taking the question more generally, let it be required to find the probability that, if P, Q, R, S be four points taken at random, one in each side of a rectangle ABCD, the quadrilateral PQRS will be less than $(\frac{1}{2}+n)$th of the rectangle, n being any fraction between 0 and $\frac{1}{2}$.

2. Put AB $=a$, BC $=c$, AS $=u$, BQ $=v$, AP $=x$, DR $=z$; then the area of the quadrilateral PQRS is

$$ac-\tfrac{1}{2}ux-\tfrac{1}{2}(a-z)(c-v)-\tfrac{1}{2}(c-u)z-\tfrac{1}{2}(a-x)v,$$

$$\text{or, } \tfrac{1}{2}\left\{ac-(u-v)(x-z)\right\};$$

and this will be *equal* to $(\frac{1}{2}\pm n)$th of ABCD, if

$$(u-v)(x-z) = \mp 2nac \ldots\ldots\ldots\ldots (\alpha).$$

From (α) we see that the area of a quadrilateral inscribed in a rectangle is *constant*, if the rectangle contained by the projections of the diagonals of the quadrilateral on the sides of the rectangle is *constant*.

Now the number of points in a straight line may be proportionally represented by its length, and thus the number of inscribed quadrilaterals whose vertices range over given parts of the four sides of the rectangle is proportional to the product of the lengths of these parts: the total number of inscribed quadrilaterals is therefore a^2c^2, and the number of those which satisfy the conditions of the problem will be given by the expression $\iiiint dx\,dz\,dv\,du$, the integral being taken between limits which we proceed to find.

3. There will evidently be as many of the required quadrilaterals formed when z is greater than x, as when z is less than x; moreover, in the latter case, the diagonal PR will meet AD produced through D, so that if Q, S be the positions of the vertices in BC, AD for which the quadrilateral is *equal* to $(\frac{1}{2}\pm n)$th of the rectangle, it will obviously be *less* if these vertices lie anywhere in BQ, DS, and *vice versâ*. It is also clear that no quadrilateral less than $(\frac{1}{2}-n)$th, or greater than $(\frac{1}{2}+n)$th, of the rectangle can be formed if $(x-z)$ is less than $2na$.

Supposing, then, z less than x, we see from (α) that u must be less or greater than v according as we take the upper or lower sign; hence the inscribed quadrilateral will be *less* than $(\frac{1}{2}-n)$th of the rectangle, or (changing the order of integration of u and v) *greater* than $(\frac{1}{2}+n)$th of it, if the variables lie between the following limits;—

$u \ldots v+\dfrac{2nac}{x-z}$ to c,	$z \ldots 0$ to $x-2na$,
$v \ldots 0$ to $c-\dfrac{2nac}{x-z}$,	$x \ldots 2na$ to a.

4. Taking the double of the integral between these limits, the number of the quadrilaterals which satisfy the conditions of the problem will be

$$\iiint 2\left(c-v-\frac{2nac}{x-z}\right)dx\,dz\,dv =$$

$$\iint c^2\left(1-\frac{2na}{x-z}\right)^2 dx\,dz =$$

$$\int c^2\left\{x-\frac{4n^2a^2}{x}+4na\log\left(\frac{2na}{x}\right)\right\}dx =$$

$$a^2c^2\left\{\tfrac{1}{2}(1-2n)(1+10n)+4n(1+n)\log(2n)\right\}.$$

5. If, therefore, we put p for the probability that the inscribed quadrilateral will be *less* than $(\frac{1}{2}-n)$th, or *greater* than $(\frac{1}{2}+n)$th of the rectangle, p will be the ratio of the number of quadrilaterals in Art. 4 to the total number (a^2c^2) of inscribed quadrilaterals; that is, we shall have

$$p = \tfrac{1}{2}(1-2n)(1+10n)+4n(1+n)\log(2n).$$

In the particular case proposed in the Question $n=\frac{1}{4}$, hence the probability $(1-p)$ that the inscribed quadrilateral will be *less* than *three-fourths* (or *greater* than *one-fourth*) of the rectangle, is

$$\frac{1}{8}+\frac{5}{4}\log_e 2.$$

This probability is $\cdot 991434$; hence, if an indefinite number of quadrilaterals be inscribed at random, not more than one in a hundred will be *less* than *one-fourth* or *greater* than *three-fourths* of the rectangle.

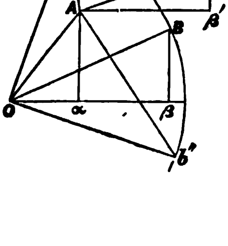
6. The *mean* or *average* of the areas of all the inscribed quadrilaterals is half the area of the rectangle. This follows at once from Art. 3, which shows that there are as many quadrilaterals *greater* than $(\frac{1}{2}+n)$th of the rectangle as there are *less* than $(\frac{1}{2}-n)$th of it. The average may be otherwise obtained by the aid of the following property:—

If a series of triangles have a common base, and their vertices lie in a given *finite* straight line which is wholly on the same side of the base (that is, if the base produced does not cut the finite straight line), the average of all the triangles thus formed is that whose vertex is at the *middle* of the finite straight line; since, for every triangle which exceeds this, there is obviously another just as much less than it.

From this it follows that the mean of all the inscribed quadrilaterals is that whose vertices are at the middles of the sides of the rectangle, and its area is therefore half that of the rectangle. The same property may be applied to find many other such averages. For instance, it shows that the mean area of all the triangles which can be inscribed in a given triangle is *one-fourth* of the given triangle, since it is the triangle formed by joining the middles of its sides.

1360 (Proposed by W. S. B. Woolhouse, F.R.A.S., F.S.S., &c., London.)—The sides of a triangle are $\alpha+i\alpha'$, $\beta+i\beta'$, where α, β, α', β' are given lines, and the included angle C is also given; determine the third side by an easy geometrical construction.

Solution by the Proposer.



Let the imaginary values be represented geometrically according to the principle of Warren and others, and in the diagram let

$$\text{O}\alpha = \alpha, \quad \alpha\text{A} = \alpha',$$
$$\text{O}\beta = \beta, \quad \beta\text{B} = \beta'.$$

Then the given sides are

$$\text{OA} = \alpha + i\alpha'.$$
$$\text{OB} = \beta + i\beta'.$$

Now if, as usual, these sides be denoted by a, b, and the required side by c, we shall have

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cos C \\ &= (a-b\cos C)^2 + (b\sin C)^2 \\ &= (a-b\cos C)^2 - (bi\sin C)^2 \\ &= (a-b\cos C-bi\sin C)(a-b\cos C+bi\sin C) \\ &= (a-b')(a-b''), \end{aligned}$$

where $b' = b(\cos C+i\sin C)$,
and $b'' = b(\cos C-i\sin C)$.

If with centre O and radius OB the arc $b'\mathrm{B}b''$ be drawn, and on both sides of OB the angles $\mathrm{BO}b'$, $\mathrm{BO}b''$ be made each equal to the given angle C; then the lines $\mathrm{O}b'$, $\mathrm{O}b''$ will represent b', b''; and if $\mathrm{A}b'$, $\mathrm{A}b''$ be joined, these lines will represent $a-b'$, $a-b''$. Hence, as the required side is a mean proportional between these last mentioned lines, it will be geometrically determined by drawing AB′ bisecting the angle $b'\mathrm{A}b''$ and taking the distance AB′ a mean proportional between $\mathrm{A}b'$ and $\mathrm{A}b''$. Therefore, if $\mathrm{A}\beta'$, $\mathrm{B}'\beta'$ be drawn respectively parallel to $\mathrm{O}\beta$, $\mathrm{B}\beta$, the third side is

$$c = \mathrm{AB}' = \mathrm{A}\beta' + i\,.\,\beta'\mathrm{B}'.$$

In the preceding Solution we recognise the following considerations:

The lines $\mathrm{O}a\mathrm{A}$, $\mathrm{O}\beta\mathrm{B}$ are regarded as coincident, because their real portions are coincident, and the unreal portions correspond in direction. The angle C may be formed by moving $\mathrm{O}\beta\mathrm{B}$ in either a positive or negative direction, so as to bring the extremity B to b' or b'', thus suggesting $\mathrm{A}b'$ or $\mathrm{A}b''$ as the third side. The true value of the third side is in direction an arithmetical mean, and in magnitude a geometrical mean, between these two lines.

[Note.—Here, as elsewhere in our columns, i is put for the square root of negative unity.—Ed.]

1443 (Proposed by Professor Sylvester, F.R.S., Royal Military Academy, Woolwich.)—Show that the locus of the centres of all the conics circumscribing a given quadrilateral is an ellipse if the quadrilateral is re-entrant, and an hyperbola if it is convex. Show further that two real parabolæ may always be drawn through the angles of any convex quadrilateral.

Solutions (1—10) by Mr. W. K. Clifford, Trinity College, Cambridge; and Mr. A. Renshaw: (11) by Dr. Hirst, F.R.S.

1. First, the locus of the centres is a conic.

Let U, V be the tangential equations of two conics through the four points; then the general equation of a conic through the points is

$$l^2p\mathrm{U} + lm\mathrm{F} + m^2q\mathrm{V} = 0 \;\;\ldots\ldots\ldots\ldots\ldots\;\; (1),$$

where p, q are the discriminants of U, V, and F is the conic touched by all the tangents to U and V at the four points of intersection; thus, if

$$\mathrm{U} \equiv ax^2+by^2+cz^2, \quad \mathrm{V} \equiv a'x^2+b'y^2+c'z^2,$$

then $$\mathrm{F} \equiv aa'(bc'+b'c)x^2+bb'(ca'+c'a)y^2 + cc'(ab'+a'b)z^2.$$

Now let (ξ, η, ζ) be a fixed straight line, and write Δ for

$$\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz};$$

then the pole of (ξ, η, ζ) with respect to (1) is

$$l^2p\Delta\mathrm{U} + lm\Delta\mathrm{F} + m^2q\Delta\mathrm{V} = 0 \;\;\ldots\ldots\;\; (2),$$

whose locus is $4pq\Delta\mathrm{U}\,.\,\Delta\mathrm{V} = (\Delta\mathrm{F})^2$, a conic section. We obtain the locus of *centres* by simply putting $\xi = \eta = \zeta$.

2. Solution by *trilinear* coordinates.

The equation to the conic is

$$\frac{l}{a} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \;\;\ldots\ldots\ldots\ldots\;\; (3),$$

subject to the condition

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \;\;\ldots\ldots\;\ldots\ldots\;\; (4).$$

The polar of a point (ξ, η, ζ) with respect to (3) is

$$\frac{a}{\xi}\left(\frac{m}{\eta}+\frac{n}{\zeta}\right) + \frac{\beta}{\eta}\left(\frac{n}{\zeta}+\frac{l}{\xi}\right) + \frac{\gamma}{\zeta}\left(\frac{l}{\xi}+\frac{m}{\eta}\right) = 0.$$

If this coincide with a fixed line $xa+y\beta+z\gamma = 0$, we must have

$$\frac{\frac{m}{\eta}+\frac{n}{\zeta}}{\xi x} = \frac{\frac{n}{\zeta}+\frac{l}{\xi}}{\eta y} = \frac{\frac{l}{\xi}+\frac{m}{\eta}}{\zeta z} = \frac{\frac{l}{\xi}+\frac{m}{\eta}+\frac{n}{\zeta}}{\frac{1}{2}(\xi x+\eta y+\zeta z)};$$

$$\therefore \frac{\frac{l}{\xi}}{-\xi x+\eta y+\zeta z} = \frac{\frac{m}{\eta}}{\xi x-\eta y+\zeta z} = \frac{\frac{n}{\zeta}}{\xi x+\eta y-\zeta z} \;\; (5).$$

Substitute these values in (4), and we have for the equation to the locus

$$\frac{\xi}{f}(-\xi x+\eta y+\zeta z) + \frac{\eta}{g}(\xi x-\eta y+\zeta z) + \frac{\zeta}{h}(\xi x+\eta y-\zeta z) = 0 \;\;\ldots\ldots\ldots\ldots\;\; (6).$$

To find the locus of *centres*, we may either consider the coordinates trilinear, and put a, b, c for x, y, z; or we may consider them triangular, and put $x=y=z=1$.

It is clear that by varying the condition (4) we may easily find the locus in other cases. Thus, for instance, "the locus of the centres of all conics passing through three given points, and touching a given straight line $(fa+g\beta+h\gamma = 0)$, is the curve of the fourth degree,

$$\sqrt{\{f\xi(-\xi+\eta+\zeta)\}} + \sqrt{\{g\eta(\xi-\eta+\zeta)\}} + \sqrt{\{h\zeta(\xi+\eta-\zeta)\}} = 0,$$

the coordinates being triangular." (Cambridge and Dublin Mathematical Journal, vol. v. p. 148.)

Another solution by trilinear coordinates has been proposed in the *Messenger of Mathematics*, vol. ii., p. 169; we give it here in order to notice one of the theorems which may be deduced from it. Consider the equations in Art. 1 as trilinear; then it may easily be proved that the locus of the pole of a line L, or $\xi x + \eta y + \zeta z = 0$, with respect

to all the conics $lU + mV = 0$, is the Jacobian of U, V, L, that is,

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz} \\ \xi, & \eta, & \zeta \end{vmatrix} = 0 \ldots\ldots\ldots\ldots(7).$$

But this is precisely the equation which has been elsewhere obtained (See Art 8 of the Solution to Quest. 1418, p. 30) as the locus of "*polar opposites*" of points in the line (ξ, η, ζ); that is, the polars of any point in this line with respect to all the conics $lU + mV$ pass through a fixed point in (7). The sides and diagonals of the quadrilateral are evidently cut harmonically by any line and its polar conic; and since the "locus of centres" is the polar conic of the line at infinity, it must coincide with the "nine-point conic," which *bisects* the sides and diagonals, besides passing through the points E, F, G (Fig. 3). That the conic (7) always *does* circumscribe the common self-conjugate triangle of U, V, may be shown by putting it in the form

$$\begin{vmatrix} a, & b, & c, \\ a', & b', & c', \\ \frac{\xi}{x}, & \frac{\eta}{y}, & \frac{\zeta}{z} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

It appears then that the nine-point conic possesses the following property: *the polars of any point of it, with respect to all the conics circumscribing the quadrilateral, are parallel.* And conversely, *all the diameters conjugate to a fixed straight line pass through a fixed point on the nine-point conic.*

The "curve of the third class," mentioned at the end of the Solution of 1418 as the envelop of lines whose polar conics degenerate, is no other than the three vertices of the common self-conjugate triangle, as readily appears from geometrical considerations. By taking, then, the discriminant of the Jacobian (7) with respect to (x, y, z), we obtain a contravariant expression for these vertices, which enables us at once to reduce two quadrics to the canonical form.

3. Construction for the *directions of the asymptotes.*

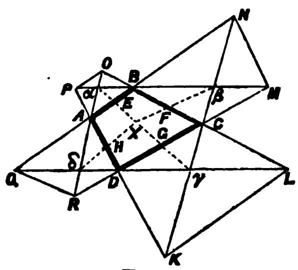


Fig. 1.

Let ABCD (Fig. 1) be the four given points. Draw any line KL parallel to AB, meeting AD, BC in K, L respectively. Then DL, CK are parallel to the asymptotes of a certain conic through A, B, C, D. This follows immediately from Pascal's Theorem. (See "Gaskin's Construction of a Conic Section, &c." p. 39, cor. 5.)

4. Construction for the *centre* of the last conic.

Describe about ABCD the parallelogram $\alpha\beta\gamma\delta$, having its sides parallel to DL, CK. Let EFGH be the bisections of the sides of ABCD.

Then αE, βF, γG, δH will meet in a point X, which is the centre of the conic.

It may be observed that KL, MN, OP, QR are respectively parallel to BA, AD, DC, CB.

5. Construction for the directions of the axes of the two *parabolæ* through the four points.

This would clearly be accomplished if we could draw KL so that CK, DL should be parallel.

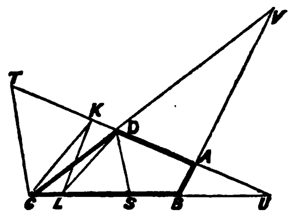


Fig. 2.

Let then the circle through DAB (Fig. 2) meet UBC in S, and the circle through ABC meet UAD in T. Take $UK^2 = US \, . \, UC$, and $UL^2 = UD \, . \, UT$; then KL is parallel to AB, and DL to CK. By treating V in the same way, we get another direction; but if the quadrilateral be re-entrant, it is easily seen that the construction fails.

This immediately determines the *species* of the conic found in arts. 1 or 2. For if two parabolæ can be described through the four points, the locus of centres must have two points at infinity, that is, it must be an hyperbola. If no parabola can be so described, the locus has no point at infinity; that is, it is an ellipse.

6. We can now easily find a construction for the locus of centres when the quadrilateral is convex.

For the asymptotes are parallel to the axes of parabolæ found in art. 5, and the locus must pass through the intersections of AB, CD, of AD, BC, and of AC, BD, each pair of lines being a conic through the four points. We have then three finite points, and two points at infinity. Construct for the centre by art. 4 (hence "if on the three sides of any triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of these parallelograms will meet in a point"); thus we can draw the axes and asymptotes; construct by Pascal's Theorem the points where the curve meets the major axis, and the thing is done. Or, of course, the length of the axis may be found more simply by performing the geometric operations indicated by the equation

$$CA^2 = CN^2 - \left(PN \, . \, \frac{CA}{CB}\right)^2,$$

P being one of the three given finite points.

7. It appears from Art. 2 that the locus will break up into two straight lines, if E or F (Fig. 3) be at infinity, that is, if two sides of the quadrilateral are parallel. This is also clear from the fact that, when a conic becomes two parallel straight lines, any point midway between them is a centre. The line at infinity is itself part of the locus when it contains two of the points A, B, C, D. If one of these four is at infinity, only one parabola (or rather two coincident parabolæ) can be drawn through them, and the locus of centres is a parabola.

When the quadrangle can be inscribed in two *different* equilateral hyperbolæ, the locus of centres is a circle; and when it can be inscribed in a circle, the locus of centres is an equilateral hyperbola, whose asymptotes are equally inclined to any pair of opposite sides, and to the two diagonals. If two equilateral hyperbolæ intersect in A, B, C, D, then AB is perpendicular to CD, AC to BD, and AD to BC, each of the points A, B, C, D, being, in fact, the intersection of perpendiculars of the triangles formed by the other three. It is easily seen that the locus of centres is, in this case, the nine-point circle of any of the four triangles ABC, BCD, CDA, DAB. See Note to Solution of Quest. 1408, p, 28.

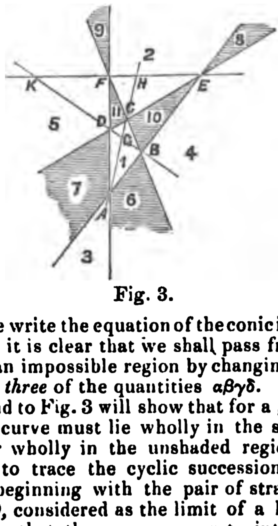


Fig. 3.

8. If we write the equation of the conic in the form $\alpha\beta=\mu\gamma\delta$, it is clear that we shall pass from a possible to an impossible region by changing the sign of *one* or *three* of the quantities $\alpha\beta\gamma\delta$. Attention to this and to Fig. 3 will show that for a given sign of μ the curve must lie wholly in the shaded regions, or wholly in the unshaded regions. We proceed to trace the cyclic succession of these curves, beginning with the pair of straight lines AB, CD, considered as the limit of a hyperbola. It is clear that these may separate into a finite hyperbola in two ways; so as to lie in the shaded regions, or in the unshaded regions. We begin with the former. One branch of the hyperbola lies entirely in (8), where also the centre is; the other branch lies in (6), (7), (10), (11), having its infinite parts in (7). The branch in (8), with the centre, moves off rapidly from E, and when the centre is at an infinite distance, we have a parabola in (6), (7), (10), (11), the infinite parts being in (7). When the parabola closes up into an ellipse, the centre reappears from the infinity of (7), and finally passes into (1). The ellipse again elongates itself, but in the direction of (6), into which the centre passes. In the limit we get another parabola, the centre going off to the infinity of (6). As the parabola merges into a hyperbola, the centre reappears from the infinity of (9), and the limit of the hyperbola is the pair of straight lines FA, FB, the centre being at F. We now pass into the unshaded regions, beginning with a small hyperbola, one branch lying in (2), (1), and (4), and the other in (3), (1), and (5). The centre is in (11), and moves down into (1). The parts of the hyperbola in (1) gradually approximate, giving as a limiting form the lines AC, BD, when the centre is at G. The branches separate again in the other direction, one lying in (2), (1), and (5), and the other in (3), (1), and (4). The centre moves from (1) into (10), and gradually approaches the point E, where the hyperbola again becomes two straight lines. This is the point from which we started. The points E, F, G lie on the *same* branch of the hyperbola which is the locus of centres, and no part of the locus lies in the regions (2), (3), (4), (5).

The re-entrant quadrangle ACEF may be treated in the same way; this case is simpler, all the conics of the series being hyperbolæ.

9. It may be worth while to notice a property of the nine-point conics of the quadrilateral faces of a tetrahedral frustum. With Prof. Sylvester's own notation, let $Oabc$ be a tetrahedron, the axes of coordinates being Oa, Ob, Oc; and let the plane $\alpha\beta\gamma$ cut off the frustum $\alpha\beta\gamma abc$. Put $4a$ for Oa, and similarly for the others; and consider the quadrilateral $ab\beta\alpha$ in the plane of xy. Its nine-point conic is easily found to be

$$\frac{x}{a\alpha}\left\{x-2(a+\alpha)\right\}=\frac{y}{b\beta}\left\{y-2(b+\beta)\right\},$$

and we draw through this a cylinder whose generating lines are parallel to the axis of z. There are three such cylinders, and they evidently have common to them the curve section

$$\frac{x}{a\alpha}\left\{x-2(a+\alpha)\right\}=\frac{y}{b\beta}\left\{y-2(b+\beta)\right\}$$

$$=\frac{z}{c\gamma}\left\{z-2(c+\gamma)\right\}.$$

We are obviously entitled to conclude that, if we take instead the polar conics of the lines in which the faces are cut by any plane, and draw cones to the points where that plane cuts the opposite edges, these three cones will have a common section.

10. The theorem stated incidentally in Art. 6 is a particular case of Brianchon's theorem. It may be put a little more generally as follows :—

Take two points P, Q, and through each of them draw three straight lines. These triads will intersect in nine points, as in the following scheme,

$$\begin{array}{c}P\\ \overbrace{\begin{array}{ccc}A&B&C\\D&E&F\\G&H&K\end{array}}\end{array}\left.\vphantom{\begin{array}{c}A\\D\\G\end{array}}\right\}Q$$

Take now three points, one from each of the P-lines, and one from each of the Q-lines, as, for instance, B, F, G. To each pair of these take the opposite diagonal of the quadrilateral, *e.g.*, to BF corresponds CE; then these three lines CE,

DK, AH will meet in a point. There are six such systems of lines.

The points P, Q may be considered as a conic inscribed in the hexagon CKHEDA, of which CE, DK, AH are the diagonals. The theorem is thus seen to be a particular case of Brianchon's. It will be found to involve also the following theorem of determinants: viz., the determinant whose constituents are the nine determinants

$$\left\|\begin{matrix} \begin{Vmatrix} C & \frac{a}{b} & A \\ c & \frac{A}{B} & a \end{Vmatrix} \\ \begin{Vmatrix} A & \frac{b}{c} & B \\ a & \frac{B}{C} & b \end{Vmatrix} \\ \begin{Vmatrix} B & \frac{c}{a} & C \\ b & \frac{C}{A} & c \end{Vmatrix} \end{matrix}\right\|$$

vanishes identically.

11. Another Solution to the Question may be obtained by showing that, according as ABCD is a convex or re-entrant quadrilateral, the pencil of conics (ABCD) does or does not contain a real pair each of which touches the line at infinity. (See Art. 5). Now the pencil (ABCD) cuts every line L in the plane in a system of points in involution, which system is determined by the intersections of any two conics, and therefore most easily by those of two of the three pairs of lines, opposite connectors of the tetrastigm, which the pencil contains. If the segments which these two pairs of lines determine upon L *partially overlap one another*, the involution has no real double points, and consequently no conics of the pencil touch L. If, on the contrary, one of these segments is *wholly within or wholly without the other*, the involution *has* real double points, and there are two conics of the pencil which touch L. Now if L be taken so that all four points A, B, C, D, lie on the same side of it, a *mere inspection of the figures* will show that if the triangle determined by any three of these points encloses the fourth, in other words if ABCD be a re-entrant quadrilateral, the involution on L will have no real double points; and that, in the other case, when ABCD is convex, such involution will always have two real double points. Taking for L, therefore, the line at infinity, the question is solved.

1455 (Proposed by G. T. SADLER, F.R.A.S.)—There are 43 balls of 4 colours, viz., 14 red, 11 blue, 12 green, and 6 white. Show in how many different ways these 43 balls may be arranged in 3 divisions, so that in the first there may be 8 red, 6 blue, 4 green, and 2 white; in the second 5 red, 4 blue, 3 green, and 3 white; and in the third 1 red, 1 blue, 5 green, and 1 white.

Solution by the PROPOSER, and ALPHA.

Writing, for shortness' sake, (n) for $1.2.3\ldots n$, the numbers (R, B, G, W, suppose) of different ways in which the red, blue, green, and white balls may be divided into three groups, containing the respective numbers required by the question, will be

$$R=\frac{(14)}{(8)(5)(1)},\qquad B=\frac{(11)}{(6)(4)(1)},$$

$$G=\frac{(12)}{(4)(3)(5)},\qquad W=\frac{(6)}{(3)(2)(1)}.$$

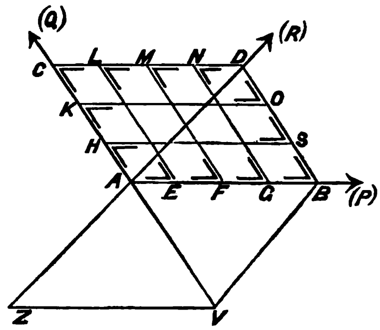
Now each of these triads may be combined with each of the others, and every combination thus obtained will satisfy the conditions of the Question; hence the total number (N) of different arrangements will be

$$N=RBGW=\frac{(14)(11)(12)}{(8)(5)^2(4)^2(3)^2}$$

$$=692250118560000.$$

1460 (Proposed by the EDITOR.)—To prove the parallelogram of forces.

Solution by the PROPOSER.



1. Let (P), (Q) be two forces, represented in magnitude and direction by the lines AB, AC; and supposing the forces to have a common measure (f), let $P=4f$, $Q=3f$, so that

$$P:Q=AB:AC=4:3.$$

Complete the parallelogram ABDC; divide AB, AC into four and three equal parts respectively in the points E, F, G, H, K; and through these points draw lines parallel to AC, AB.

Now, supposing these lines to be rigidly connected with one another, it is evident that the state of the system will not be affected if we introduce at H and E, the opposite angles of the *rhombus* HE, two forces each equal to f, acting along the sides of that rhombus in the directions HA, HS, EA, EL (as indicated in the figure); for the pairs of forces at H and E will obviously give two equal resultants which act in

opposite directions along the bisectors of the angles H and E, that is, along the diagonal HE.

In like manner we may introduce two forces, each equal to f, at the opposite angles K and F, C and G, L' and B, M and S, N and O, of the rhombuses KF, CG, LB, MS, NO.

Now the forces acting along the lines AB, HS, KO, AC, EL, FM, GN neutralize each other; and thus the system is reduced to a force of $4f$ or (P) acting along CD, and a force of $3f$ or (Q) acting along BD, which may both be supposed to be applied at the point D. Hence the two forces (P) and (Q), applied at A, may be replaced by two forces (P) and (Q), applied at D, without altering the effect of these forces; therefore the resultant (R) must pass through A and D, that is, it must act along the *diagonal* AD of the parallelogram.

The proof holds for *any two commensurable* forces, and may, by a *reductio ad absurdum*, be extended to *incommensurable* forces; hence, in all cases, the diagonal represents the resultant in *direction*.

2. Next, the diagonal will also represent the resultant in *magnitude*.

In DA produced take AZ of such a length as to represent the resultant (R) in *magnitude*; then the three forces AB, AC, AZ keep the point A in equilibrium. Complete the parallelogram BAZV; then its diagonal AV will be the *direction* of the resultant of AB, AZ (by Art. 1); hence AV is in the same straight line with AC, and is therefore parallel to BD, so that AVBD is a parallelogram. We have, then, from the parallelograms ZB, VD,

$$AZ = VB = AD.$$

Hence the diagonal AD represents the resultant both in *magnitude* and direction.

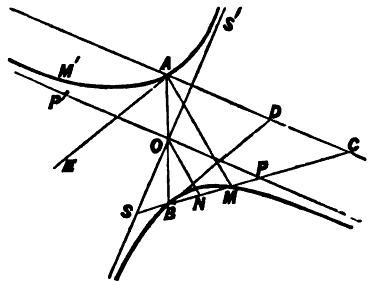
[We have taken the foregoing proof of the parallelogram of forces from Sturm's *Cours de Mécanique de l'Ecole Polytechnique;* and our object in proposing the Question was to have an opportunity of publishing Sturm's proof, which, we think, merits attention by its simplicity.]

1470 (Proposed by W. J. Miller, B.A., Mathematical Master, Huddersfield College.)—The base of an isosceles triangle is on a fixed straight line, and the equal sides pass through two fixed points, one of which is on the given line; find *geometrically* the locus of the vertex, and deduce therefrom a solution of the following problem :—

"If a luminous point be reflected by a small plane mirror, so as to be seen by the eye in a given position, and the mirror move in such a way that the luminous point always appears to be upon a given conical surface, of which the point is the vertex, and a line through the eye the axis; find the form of the surface upon which the small mirror must always be situated."

Solutions by Archer Stanley; the Rev. R. Townsend, M.A., Fellow and Tutor of Trinity College, Dublin; and the Proposer.

1. If A be the fixed corner, and AC the fixed



direction of the base of the isosceles triangle AMC, one of whose sides always passes through a second fixed point B, it is manifest that, as the vertex M moves, the sides AM, BC will rotate in opposite directions through equal angles; in other words, that these sides will be corresponding rays of two equal and opposite pencils whose centres are at A and B. The locus of the intersection M of such rays is, therefore, a rectangular hyperbola passing through A and B, having one asymptote POP' parallel; and the other SOS' perpendicular, to the fixed line AC.

Moreover the tangents, AE and BD, at A and B, to this hyperbola, are parallel to one another, being sides of isosceles triangles which have their vertices at A and B respectively. The centre O of the hyperbola is, therefore, the middle of AB.

2. Otherwise, through O, the middle of AB, draw ON parallel to AM, POP' parallel to AC, and SOS' perpendicular to AC; then the triangles ONP, ONS are isosceles, therefore PN = NO = NS; also MN = NB, because AO = OB; hence, in every position of M, we shall have PM = BS; the locus of M is therefore a rectangular hyperbola, of which AB is a diameter, O the centre, and POP', SOS' the asymptotes.

3. If a small mirror, having its reflecting plane always perpendicular to AC, be made to describe either of the hyperbolic arcs AM', BM, the image of A, to an eye at B, would clearly describe the fixed line AC. And if the whole figure be turned around AB, the surface of rotation generated by the hyperbola will be the locus of the small mirror M, which, to the eye at B, always throws the image of A upon the right cone generated by AC.

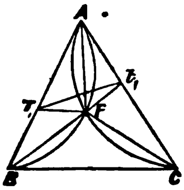
1456 (Proposed by Mr. W. Hopps, Hull.)—The points $T_1, T_2, \ldots T_n$, and $t_1, t_2, \ldots t$ are so situated in the sides AB, AC of the triangle ABC, that

$$\frac{BT_1}{T_1A} = \frac{At_1}{t_1C};\ \frac{BT_2}{T_2A} = \frac{At_2}{t_2C};\ \ldots\ \frac{BT_n}{T_nA} = \frac{At_n}{t_nC}.$$

Find the position of a point F, so that, drawing $FT_1, FT_2, \ldots FT_n$, and $Ft_1, Ft_2, \ldots Ft_n$, we may have

$$\frac{FT_1 \cdot FT_2 \ldots FT_n}{Ft_1 \cdot Ft_2 \ldots Ft_n} = \frac{(AB)^n}{(AC)^n}.$$

Solution by the PROPOSER; R. TUCKER, M.A.; and Mr. J. WILSON.



1. On AB, AC construct circular segments, each containing an angle equal to the supplement of BAC: the intersection (F) of the arcs of these segments is the point required. For, draw FA, FB, FC, T_1t_1; then

$$\angle BAF + FAC = ACF + FAC, \quad \therefore BAF = ACF.$$

Hence the triangles BAF, ACF are similar;

$$\therefore BF : FA = AB : AC \quad \ldots\ldots\ldots\ldots(1).$$

$$\text{But } BT_1 : T_1A = At_1 : t_1C;$$

$$\therefore BT_1 : BT_1 + T_1A = At_1 : At_1 + t_1C;$$

$$\text{hence } BT_1 : At_1 = AB : AC \ldots\ldots(2).$$

Consequently, from (1) and (2) we get

$$BF : FA = BT_1 : At_1.$$

But $\angle ABF = CAF$, therefore (Euc. vi. 6) the triangles BFT_1, AFt_1 are similar,

$$\therefore FT_1 : Ft_1 = BT_1 : At_1 = AB : AC \ldots(3).$$

Similarly it may be proved that

$$FT_2 : Ft_2 = AB : AC \quad \ldots\ldots\ldots\ldots(4),$$

$$\text{and } FT_n : Ft_n = AB : AC \quad \ldots\ldots\ldots\ldots(5).$$

And by compounding (3), (4), and (5) we get

$$\frac{FT_1 \,.\, FT_2 \ldots FT_n}{Ft_1 \,.\, Ft_2 \ldots Ft_n} = \frac{(AB)^n}{(AC)^n}.$$

2. Since the triangles BAF, ACF are similar,

$$FB : FA = FA : FC,$$

$$\therefore FA^2 = FB \,.\, FC.$$

3. The triangles BFT_1, AFt_1 being similar, we have $\angle AFt_1 = BFT_1$,

$$\therefore T_1Ft_1 = AFB = \text{supplement of } BAC.$$

Whence it is manifest that the circles through AT_1t_1, $AT_2t_2 \ldots AT_nt_n$ pass through F.

4. A new method of solving a known problem is contained in the preceding investigation; for, in the *determined* triangle FAB, the side AB and the opposite angle AFB are given, and

$$BF : FA = AB : AC = \text{a given ratio.}$$

5. By the property proved in the first Solution of Question 1440 (see p. 37) it is easy to show that the lines T_1t_1, $T_2t_2 \ldots T_nt_n$ all touch a parabola to which the sides AB, AC are tangents at B, C; and the focus (F) of this parabola is the point required. For, calling the points of contact $V_1, V_2 \ldots V_n$, we have, by known properties of the parabola (See art. 7 of the Solution of Question 1402, p. 20)

$$(FT_1)^2 : (Ft_1)^2 = FB \,.\, FV_1 : FC \,.\, FV_1$$

$$= FB : FC = (AB)^2 : (AC)^2,$$

$$\therefore FT_1 : Ft_1 = AB : AC;$$

and the rest follows as in art. 1.

1165 (Proposed by EXHUMATUS.)—A coin is dropped over a grating composed of parallel equidistant wires in a horizontal plane; find the chance that it will go through without striking.

Solution by W. J. MILLER, B.A., Mathematical Master, Huddersfield College.

1. Whatever rotatory motion the coin may have when it reaches the grating may be supposed to affect the chances *for* and *against* striking in an equal degree; the probability of striking will then be the same as if the coin were constrained to move parallel to itself during the very small space of time which intervenes between its *reaching* and *quitting* the plane of the wires.

All the different positions of the coin with respect to the wires will, therefore, be determined by combining its positions, when the centre is at a fixed point in the plane, with the different positions of the centre, and these latter may obviously be obtained by supposing the centre to to move perpendicularly from one wire *half-way* towards the next.

2. In order to estimate the number of different positions of the coin when its centre is fixed, imagine an axis to pass, perpendicularly to the coin, through its centre; then it is clear that, for every position of the centre, there will be as many different positions of the coin (so far as regards its striking the wires or not) as there are different directions of the axis, since its revolutions round the axis in no wise affect its position for striking the wires.

Moreover, if all possible positions of the coin be supposed equally probable, the axial directions must be equally distributed throughout space; the number of such directions will therefore be proportional to the area traced out by the axis on the surface of any sphere concentric with the coin, that is, proportional to the portion of spheric surface passed over by the *pole* of the coin.

3. Refer the axial positions to the sphere of the heavens, and let θ be the *zenith distance* of the pole of the coin, and ϕ its *azimuth*, estimated from the direction of the wires; also let a be the radius of the coin, $2c$ the distance between two consecutive wires, and x the distance of the centre of the coin from the nearest wire.

Then the probability that the pole of the coin will lie on an element of the spheric surface between (θ, ϕ) and $(\theta+\Delta\theta, \phi+\Delta\phi)$ is the ratio $(\sin\theta\,\Delta\theta\,\Delta\phi : \frac{1}{2}\pi)$ of that element to the octant of the surface between $(\theta=\phi=0)$ and $(\theta=\phi=\frac{1}{2}\pi)$ which corresponds to the total number of different positions of the pole; also $(\Delta x : c)$ is the probability that the centre will fall between the distances $(x, x+\Delta x)$ from the nearest wire.

Hence the probability (p) that the coin *will* strike the grating, or (q) that it *will not*, is the limit of $\frac{2}{\pi c}\Sigma(\sin\theta\,\Delta x\,\Delta\phi\,\Delta\theta)$; that is, we shall have

$$\tfrac{1}{2}\pi cp \text{ (or } \tfrac{1}{2}\pi cq) = \iiint \sin\theta\, dx\, d\phi\, d\theta \ldots\ldots(A),$$

the integrals being taken between limits which may be determined by the following method;—

4. The inclination of the coin to the plane of the wires is θ, and its projection on this plane is an ellipse whose semi-axes are a and $a\cos\theta$; also ϕ is the inclination of the *minor* axis to the wires, or of the *major* axis to the line (x) perpendicular to the wires. When, therefore, the ellipse *just touches* one of the wires, x is a perpendicular from the centre on a tangent, and ϕ the inclination of this perpendicular to the the major axis; hence, putting $x = a\cos\chi$ (so long as x does not exceed a) we have, by a known property of the ellipse,

$$x^2 = a^2\cos^2\chi = a^2(1-\sin^2\theta\sin^2\phi);$$

the equation of limits is, therefore,

$$\sin\theta\sin\phi = \sin\chi \ldots\ldots\ldots\ldots\ldots\ldots(\text{B}).$$

It will be found convenient to effect the final integration of (A) by introducing a subsidiary variable ψ, connected with ϕ by the relation $a\cos\phi = c\sin\psi$, or, putting $c = a\cos\alpha$ (when c is less than a), $\cos\phi = \cos\alpha\sin\psi$. Hence, if β be an angle such that $\sin\beta\sin\phi = \sin\chi$, we see from (B) that the coin *will not strike the grating* while the variables are between the following limits:—

$$\left.\begin{array}{l|l} \theta \ldots \beta \text{ to } \tfrac{1}{2}\pi & \phi \ldots \alpha \text{ to } \tfrac{1}{2}\pi \\ \chi \ldots \phi \text{ to } \alpha & \psi \ldots \tfrac{1}{2}\pi \text{ to } 0 \end{array}\right\} \ldots(\text{C}).$$

5. The value of $\frac{1}{2}\pi cq$ is therefore given by the following integration, from (A), the limits being those in (C):—

$$\iiint a\sin\theta\, d\phi\, d(\cos\chi)\, d\theta =$$

$$\iint a\,(1-\operatorname{cosec}^2\phi\sin^2\chi)^{\frac{1}{2}}\, d\phi\, d(\cos\chi) =$$

$$\iint a\operatorname{cosec}\phi\,(\cos^2\chi-\cos^2\phi)^{\frac{1}{2}}\, d\phi\, d(\cos\chi) =$$

$$\int \tfrac{1}{2}a\operatorname{cosec}\phi\,(\cos^2\alpha\cos\psi-\cos^2\phi\log\cot\tfrac{1}{2}\psi)\, d\phi;$$

whence, in terms of ψ, we have

$$\pi q = \int_0^{\frac{1}{2}\pi} \frac{\cot^2\psi-\cos\psi\log\cot\frac{1}{2}\psi}{\sec^2\alpha\cot^2\psi+\tan^2\alpha}\, d\psi \ldots(\text{D}).$$

6. To find the value of the definite integral (D), put it in the form $\pi q = \text{I}_1+\text{I}_2-\text{I}_3+\text{I}_4$,

$$\text{where } \text{I}_1 = \int_0^{\frac{1}{2}\pi} \frac{\cos^2\psi\, d\psi}{\sec^2\alpha-\sin^2\psi};$$

$$\text{I}_2 = \int_0^{\frac{1}{2}\pi} \cos\psi\log\cot\tfrac{1}{2}\psi\, d\psi;$$

$$\text{I}_3 = \int_0^{\frac{1}{2}\pi} \sec\psi\log\cot\tfrac{1}{2}\psi\, d\psi;$$

$$\text{I}_4 = \int_0^{\frac{1}{2}\pi} \frac{(\sec\psi-\cos\psi)\log\cot\frac{1}{2}\psi\, d\psi}{(\sec^2\alpha-\sin^2\psi)\cot^2\alpha};$$

$$\text{then } \text{I}_1 = \left\{\psi-\sin\alpha\cot^{-1}\left(\frac{\cot\psi}{\sin\alpha}\right)\right\}_{\psi=0}^{\psi=\frac{1}{2}\pi}$$

$$= \frac{\pi}{2}(1-\sin\alpha);$$

$$\text{I}_2 = \left\{\psi+\sin\psi\log\cot\tfrac{1}{2}\psi\right\}_{\psi=0}^{\psi=\frac{1}{2}\pi} = \frac{\pi}{2};$$

also, putting $\cot\frac{1}{2}\psi = e^z$, $2\cos 2\alpha = s$, we have

$$\text{I}_3 = \int_0^\infty 2\,(e^{-z}+e^{-3z}+e^{-5z}+\ldots)\, z\, dz$$

$$= 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right) = \frac{\pi^2}{4};$$

$$\text{I}_4 = \int_0^\infty \frac{2\,(2-s)\,e^{-3z}z\, dz}{(1-e^{-2z})\,(1-se^{-2z}+e^{-4z})}$$

$$= 2(2-s)\int_0^\infty z\, dz\left\{e^{-3z}+(1+s)\,e^{-5z}+\ldots\right\}$$

$$= 2(2-s)\left\{\frac{1}{3^2}+\frac{1+s}{5^2}+\frac{s(1+s)}{7^2}+\ldots\right\}.(\text{E}).$$

Hence, putting S for the bracketed series in (E),

$$p = \tfrac{1}{2}\sin\alpha+\frac{\pi}{4}-\frac{2}{\pi}(2-s)\,\text{S} \ldots\ldots\ldots(\text{F}).$$

7. This expression (F) gives the general values of the probabilities, (p) of striking, or ($q = 1-p$) of not striking, when the distance between two consecutive wires does not exceed the diameter of the coin. If, for example, the distance between the wires is *equal to the radius of the coin*,

$$a = 2c,\quad \alpha = \tfrac{1}{3}\pi,\quad s = -1, \text{ and then}$$

$$\text{S} = \frac{1}{3^2}+\frac{1}{9^2}+\frac{1}{15^2}+\ldots = \frac{\pi^2}{72};$$

$$\therefore\ p = \frac{\pi}{6}+\frac{\sqrt{3}}{4}, \text{ or } p = \cdot 9566,\ q = \cdot 0434;$$

wherefore in this case the odds are about 22 to 1 in favour of striking.

8. When the distance between the wires is *equal to the diameter of the coin*, $a = c$, $\alpha = 0$, $\text{S} = 0$,

$$\therefore\ p = \frac{\pi}{4},\ q = 1-\frac{\pi}{4}; \text{ or } p : q = 11 : 3 \text{ nearly};$$

that is, the odds are about 11 to 3 in favour of striking.

9. When the distance between the wires is *greater than the diameter of the coin*, it is clear that the coin cannot strike unless its centre reaches the plane of the grating at a distance less than its radius from the nearest wire; but the probability that it will fall within this distance is ($a : c$), since it is equally likely to fall at any distance

less than c; moreover, if the centre reaches the grating at a distance less than a from the nearest wire, the probability of striking is $\frac{1}{2}\pi$, by Art. 8; hence in this case the probabilities are

$$p = \frac{\pi a}{4c}, \quad q = 1 - \frac{\pi a}{4c}.$$

For example, when the distance between the wires is *equal to the circumference of the coin*,

$$p = \frac{1}{4}, \quad q = \frac{3}{4}; \quad \text{or } p:q = 1:3;$$

that is, the odds are 3 to 1 against striking.

10. Another Solution to the Question may be obtained in the following manner:—

Consider the grating as an indefinite plane, and the wires as straight lines drawn on it; and suppose an ellipse to be thrown on the plane.

Then, ϕ being the inclination of the minor axis to the parallels, and x the distance of the centre from the nearest parallel, when, with this value of ϕ, the ellipse just touches that parallel; the probability that the ellipse, if thrown at random, will rest on one of the lines, is the limit of

$$\Sigma\left(\frac{x}{c} \cdot \frac{\Delta\phi}{\frac{1}{2}\pi}\right), \text{ that is } \frac{2}{\pi c}\int_0^{\frac{1}{2}\pi} x\, d\phi \text{ ....(G)};$$

for $(\Delta\phi : \frac{1}{2}\pi)$ is the probability that the inclination will be between $(\phi, \phi+\Delta\phi)$, and $(x:c)$ is the probability of resting on one of the lines with this inclination. Now suppose this ellipse to be the *projection of the coin* (see Art. 4); then, putting in (G) for x its value $a(1-\sin^2\theta\sin^2\phi)^{\frac{1}{2}}$, the probability that the coin will strike the grating is

$$\frac{2a}{\pi c}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta\sin^2\phi)^{\frac{1}{2}}\sin\theta\, d\theta\, d\phi =$$

$$\frac{a}{2c} + \frac{a}{\pi c}\left(I_3 - I_2\right) = \frac{\pi a}{4c}, \text{ as found in Art. 9.}$$

This Solution applies to the case in which the distance between the wires is not less than the diameter of the coin; but the method may be easily extended to the other cases of the problem.

11. In (G), x is the length of the perpendicular from the centre on a tangent to the ellipse, and ϕ the inclination of this perpendicular to the major axis; hence, by a well known formula in rectification, the *integral* in (G) expresses the length of a quadrant of the curve. If then l be the perimeter of the ellipse, and l' $(=2\pi c)$ that of the greatest circle that can be placed between the parallels, the probability that the ellipse will rest on one of these parallels is $(l : l')$. The same may be proved in like manner of any complete oval which has no singular points, and whose greatest diameter does not exceed $2c$ (see Ex. 11 of the unsolved Questions in Todhunter's Int. Calc., p. 290, 2nd ed.); and in my Solution of Quest. 1274 (*Educational Times* for Aug. 1862) it is shown that there is the same probability of crossing when a *regular polygon* is thrown at random on the plane.

12. The number of positions of the coin, for an inclination θ to the plane, varies as $\sin\theta$, by Art. 3; hence, λ being a constant, the sum of the inclinations in all these positions is

$$\int_0^{\frac{1}{2}\pi}\lambda\theta\sin\theta\, d\theta = \lambda(\sin\theta - \theta\cos\theta)_{\theta=0}^{\theta=\frac{1}{2}\pi} = \lambda,$$

and the number of such positions is

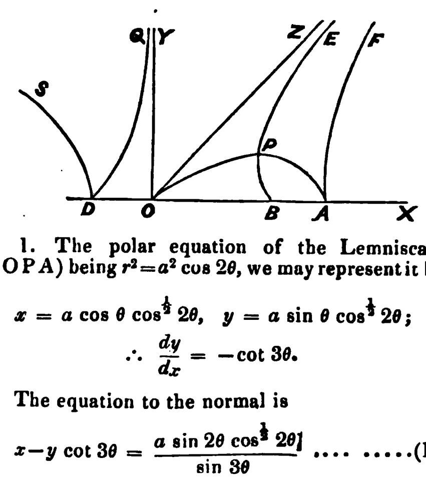
$$\int_0^{\frac{1}{2}\pi}\lambda\sin\theta\, d\theta = \lambda;$$

therefore the *mean* or *average* of all the inclinations with which the coin may reach the plane is the unit of circular measure, or the angle (57° 17′ 44·8″) subtended by an arc equal to the radius.

In this way we may see that "if the orbits of comets be supposed to be equally distributed through space, their mean inclination to the plane of the ecliptic is the angle subtended by an arc equal to the radius." (Todhunter's Int. Calc., p. 290, Ex. 8.)

1407 (Proposed by R. Tucker, M.A.)—Find the equations to the *Evolutes* of (1) the *Lemniscate*, and (2) the *Cissoid*; and trace them.

Solution by the Proposer; and Alpha.



1. The polar equation of the Lemniscate (OPA) being $r^2 = a^2\cos 2\theta$, we may represent it by

$$x = a\cos\theta\cos^{\frac{1}{2}}2\theta, \quad y = a\sin\theta\cos^{\frac{1}{2}}2\theta;$$

$$\therefore \frac{dy}{dx} = -\cot 3\theta.$$

The equation to the normal is

$$x - y\cot 3\theta = \frac{a\sin 2\theta\cos^{\frac{1}{2}}2\theta}{\sin 3\theta} \text{ ....(1)}.$$

Differentiating (1) with respect to θ, we obtain

$$y = -\frac{2a\sin^3\theta}{3\cos^{\frac{1}{2}}2\theta} \text{ ....(2)}$$

$$\therefore x = \frac{2a\cos^3\theta}{3\cos^{\frac{1}{2}}2\theta} \text{ ....(3)}.$$

Eliminating θ from (2), (3), the *Evolute* is

$$9(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{4}{3}} - y^{\frac{4}{3}}) = 4a^2 \text{ ....(4)}.$$

Now $\frac{dy}{dx} = \tan 3\theta$; $\frac{dx}{d\theta} = -\frac{a \sin 2\theta \cos 3\theta}{3 \cos^{\frac{3}{2}} 2\theta}$;

$$\frac{d^2y}{dx^2} = -\frac{9 \cos^{\frac{3}{2}} 2\theta}{a \sin 2\theta \cos^3 3\theta};$$

hence x has its least value $(= \frac{1}{4} a \sqrt{6})$ when $\theta = 30°$, and $y = -\frac{1}{12} a \sqrt{2}$.

When $\theta = 0$, $x = \frac{2}{3}a$, $y = 0$, $\frac{dy}{dx} = 0$;

when $\theta = 45°$, $x = \infty$, $y = -\infty$;

hence the asymptotes of the *Evolute* coincide with those of the rectangular Hyperbola corresponding to the Lemniscate. In the *right hand* part of the diagram, AF is the rectangular Hyperbola, OPA the corresponding Lemniscate, BPE its Evolute, and OZ the asymptote. Similar portions lie below OX and to the left of OY.

2. Take for the equations to the Cissoid

$$x = 2a \sin^2 \theta, \quad y = 2a \sin^2 \theta \tan \theta;$$

then, if $\tan \theta = t$, the equation to the normal is

$$2x + (t^2+3)\,ty = 2(t^2+2)\,at^2 \ldots\ldots\ldots\ldots (5).$$

Differentiating (5) with respect to t, we obtain

$$y = \tfrac{8}{3}at, \quad x = -\tfrac{1}{3}(t^2+6)\,at^2 \ldots\ldots\ldots\ldots (6);$$

hence eliminating t, the equation of the Evolute is

$$\tfrac{9}{64} y^2 + 3a^2 = (9a^4 - 3a^3x)^{\frac{1}{2}} \ldots\ldots\ldots\ldots (7).$$

From (7), $\frac{9}{32} y \frac{dy}{dx} = -\frac{3a^3}{2(9a^4-3a^3x)^{\frac{1}{2}}}$.

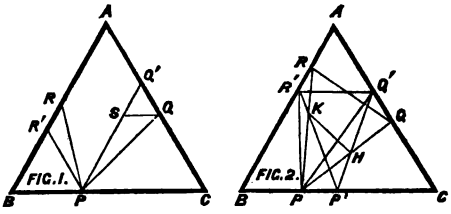
The curve lies wholly to the left of the origin, cuts the axis at right angles, and is ultimately parallel to the axis. In the *left hand* part of the diagram, D is the origin, DQ is the Cissoid, OY its asymptote, and DS its Evolute. An exactly similar portion lies below DO.

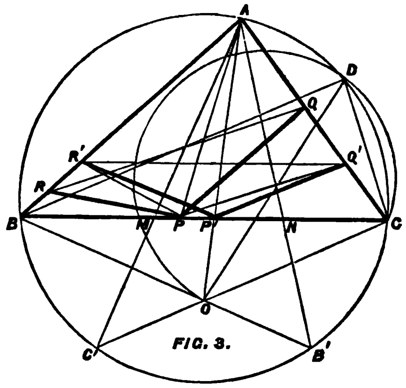
1428 (Proposed by the EDITOR.)—From a point in the base of a triangle two straight lines are drawn containing a given angle, and forming a quadrilateral with the other two sides of the triangle. It is required to prove that

(α) Of all the quadrilaterals which have a common vertex at the *same* point within the vertical angle of the triangle, that of which the sides passing through this point are equal to each other is a *maximum*, a *minimum*, or *neither* a maximum nor a minimum, according as the given angle is *less* than, *greater* than, or *equal* to, the supplement of the vertical angle of the triangle.

(β) Of all these maximum or minimum quadrilaterals which have their vertices at *different* points in the base, the *least maximum* or the *greatest minimum* is that whose equal sides make equal angles with the base.

Solution by the PROPOSER.





1. Let ABC be the given triangle, P a point in the base BC, and QPR the given angle, whose sides form, with the *indefinitely produced* lines AB, AC, the quadrilateral AQPR, which may be either *convex*, *reentrant*, or *intersecting*. Let β, γ be the perpendiculars from P on AC and AB respectively; u the sum of double the areas of the triangles BPR and CPQ; K the sum of the given angles A and P; $\theta = \angle$ PQC, and $\phi = \angle$ PRB.

Then $\theta + \phi = A + P = K$, $\therefore \frac{d\phi}{d\theta} = -1$;

$\beta \operatorname{cosec} C + \gamma \operatorname{cosec} B = a$, $\therefore \frac{d\gamma}{d\beta} = -\frac{\sin B}{\sin C}$;

$$u = \beta^2 (\cot \theta + \cot C) + \gamma^2 (\cot \phi + \cot B);$$

$$\frac{du}{d\theta} = \gamma^2 \operatorname{cosec}^2 \phi - \beta^2 \operatorname{cosec}^2 \theta;$$

$$\frac{d^2u}{d\theta^2} = 2(\beta^2 \cot \theta \operatorname{cosec}^2 \theta + \gamma^2 \cot \phi \operatorname{cosec}^2 \phi);$$

$$\frac{du}{d\beta} = \frac{2}{\sin C} \left\{ \frac{\beta \sin(\theta + C)}{\sin \theta} - \frac{\gamma \sin(\phi + B)}{\sin \phi} \right\};$$

$$\frac{d^2u}{d\beta^2} = \frac{2}{\sin^2 C} \left\{ \frac{\sin C \sin(\theta + C)}{\sin \theta} + \frac{\sin B \sin(\phi + B)}{\sin \phi} \right\}$$

$$\frac{d^2u}{d\beta d\theta} = \frac{-2}{\sin C} \left\{ \frac{\beta \sin C}{\sin^2 \theta} + \frac{\gamma \sin B}{\sin^2 \phi} \right\}.$$

2. When P is a *fixed* point (Fig. 1), β and γ are constant, and the maximum or minimum values of u are given by

$\frac{du}{d\theta} = 0$, or $\beta \operatorname{cosec} \theta = \gamma \operatorname{cosec} \phi$, or PQ = PR;

and then $\frac{d^2u}{d\theta^2}=\frac{2\beta\gamma\sin(\theta+\phi)}{\sin^2\theta\sin^2\phi}=\frac{2\beta\gamma\sin K}{\sin^2\theta\sin^2\phi}$,

which, for any point within the angle A, is *positive*, *negative*, or *zero*, according as K is *less than*, *greater than*, or *equal to* two right angles.

It follows, therefore, that u is a *minimum* when P is *less* than the supplement of A, and a *maximum* when it is *greater*. Moreover, when P is *equal* to the supplement of A (or $K=\pi$), we shall have $\theta+\phi=\pi$, and $\operatorname{cosec}\theta=\operatorname{cosec}\phi$; hence in this case PQ cannot be equal to PR unless $\beta=\gamma$, that is, unless P is in the *bisector* of A, and then we shall always have $u=\beta^2(\cot B+\cot C)$, and $PQ=PR$; that is, for all values of θ and ϕ, the area of the quadrilateral is then *constant* ($=\beta^2\cot\frac{1}{2}A$), and the sides through P are equal to each other.

We thus obtain the theorems in (α), since the quadrilateral is a *maximum* when u is a *minimum*, and *vice versâ*; for if q be double the area of the quadrilateral, and t that of the given triangle, each affected with its proper sign, we shall always have

$$q+u=t=a\ constant.$$

When P is a fixed point, the position and magnitude of the maximum or minimum quadrilateral are given by the following expressions:—

$$\cot\theta=\frac{\gamma}{\beta}\operatorname{cosec}K+\cot K,$$

$$\cot\phi=\frac{\beta}{\gamma}\operatorname{cosec}K+\cot K,$$

$$CQ=\frac{\beta\sin(B-P)+\gamma\sin C}{\sin K\sin C},$$

$$BR=\frac{\gamma\sin(C-P)+\beta\sin B}{\sin K\sin B},$$

$$u=\frac{\beta^2\sin(B-P)}{\sin K\sin C}+\frac{2\beta\gamma}{\sin K}+\frac{\gamma^2\sin(C-P)}{\sin K\sin B}\ \ldots.(I).$$

3. We have already found the *partial* maxima and minima values of the quadrilateral, so far as they depend on the variations of θ (and ϕ) alone, when β is constant and the point P is *fixed*; and when P is a *variable* point in BC, we shall obtain the *total* maxima or minima values, if any such exist, by combining with the former those which arise from the variations of β (and γ); but these latter maxima or minima are given by

$\frac{du}{d\beta}=0$, whence $\theta+C=\phi+B$; and then

$$\theta=\tfrac{1}{2}(\pi+P)-C,\ \phi=\tfrac{1}{2}(\pi+P)-B,$$

$$\text{and }\angle QPC=RPB=\tfrac{1}{2}(\pi-P).$$

Further, for a total maximum or minimum, Lagrange's condition must be satisfied; but putting

$$\lambda=\sin B\cos(C-\tfrac{1}{2}P)+\sin C\cos(B-\tfrac{1}{2}P),$$

we find, for the foregoing values of θ and ϕ,

$$\frac{d^2u}{d\theta^2}\frac{d^2u}{d\beta^2}-\left(\frac{d^2u}{d\theta\,d\beta}\right)^2=\frac{-8\lambda\beta\gamma\sin\frac{1}{2}P\operatorname{cosec}^2C}{\cos^2(B-\frac{1}{2}P)\cos^2(C-\frac{1}{2}P)},$$

an essentially *negative* expression for any position of P within the angle BAC. Hence no such position gives a total maximum or minimum; in fact, the partial maximum for the variations of θ corresponds to the partial minimum for the variations of β, and *vice versâ*.

4. The results arrived at in the last article may be obtained by a somewhat different method. Consider the expression (I) for u; it is required to find for what values of β (and γ) u becomes a maximum or a minimum. In other words, we have to solve the following problem.

In a given triangle (ABC) to inscribe an *isosceles* triangle (PQR) of given species, so that the quadrilateral (AQPR) whose diagonal (QR) is the base of the inscribed triangle may be a maximum or a minimum.

Now from (I) we have

$$\frac{du}{d\beta}=\frac{2\sin\frac{1}{2}P}{\sin K\sin C}\left\{\gamma\cos(C-\tfrac{1}{2}P)-\beta\cos(B-\tfrac{1}{2}P)\right\}$$

$\frac{d^2u}{d\beta^2}=-2\lambda\sin\frac{1}{2}P\operatorname{cosec}^2C\operatorname{cosec}K$; and

$\frac{du}{d\beta}=0$ gives $\frac{\beta}{\gamma}=\frac{\cos(C-\frac{1}{2}P)}{\cos(B-\frac{1}{2}P)}=\frac{\sin\theta}{\sin\phi}$,

whence $\theta-\phi=B-C$, since $\theta+\phi=A+P$,

$\therefore\ \theta+C=\phi+B$, or $\angle QPC=RPB$.

Also $\frac{d^2u}{d\beta^2}$ is positive or negative according as K is greater or less than two right angles; hence for the variations of β, when $\angle QPC=RPB$, u is a *maximum* if the given angle P is *less* than the supplement of A, and a *minimum* if it is *greater*; which, with what is proved in Art. 2, shows that when PQ and PR are equal to each other and make equal angles with BC, the quadrilateral is the *least maximum* or the *greatest minimum* according as P is less or greater than the supplement of A.

The quadrilateral is then given by

$$\theta=\tfrac{1}{2}(\pi+P)-C,\ \phi=\tfrac{1}{2}(\pi+P)-B,$$

and $\beta\sin\phi=\gamma\sin\theta$, which may be considered the trilinear equation of the line AP.

5. From Art. 2, the length of CQ is

$$\left\{a\sin B-2\beta\sin\tfrac{1}{2}P\operatorname{cosec}C\cos(B-\tfrac{1}{2}P)\right\}\operatorname{cosec}K,$$

which, as β increases, clearly decreases or increases according as K is less or greater than two right angles; hence we see that for the maximum or minimum quadrilaterals which have their vertices at different points in BC, as the point P moves in the direction CB, the point Q will move in the direction CA or AC according as the given angle P is greater or less than the supplement of A. This shows the relative positions of the quadrilaterals (Figs. 2, 3).

6. We shall next show how to construct any one of these quadrilaterals geometrically.

On the opposite side of BC from A construct (Fig. 3) an isosceles triangle having its vertical

angle BOC equal to the constant angle P of the quadrilateral; join AO, cutting BC in P′, and through P′ draw P′Q′, P′R′ parallel to OC, OB; then AQ′P′R′ will be the *least* of all the *maximum* quadrilaterals which have a constant angle *less* than the supplement of A, or in the other case the *greatest* of all the *minimum* quadrilaterals which have an angle *greater* than the supplement of A. For it is clear that P′Q′, P′R′ are both *equal* to each other and equally inclined to BC.

Again, to construct the maximum or minimum quadrilateral which shall have its vertex at another point P in BC. Join AP, and on OC construct a circular segment containing an angle equal to PAC, and cutting in D the circle through ABC; join DB, DO, DC, and make the angles APQ, APR respectively equal to DOC, DOB. Then (Euc. vi. 18, 20) the quadrilateral AQPR is similar to DCOB, therefore the triangle PQR is similar to OBC, that is, it is an *isosceles* triangle having its vertical angle equal to the given angle; hence AQPR is the quadrilateral required. When P is at P′, the circular segment on OC passes through A, and this segment decreases as the angle CAP increases; hence as P moves from P′ towards B, the point D must move in the direction AB or AC according as O is *without* or *within* the circle ABC, or according as the given angle is less or greater than the supplement of A. And OD, QR evidently revolve in *opposite* directions; hence we obtain from *geometrical* considerations the result deduced from our *analysis* in Art. 5.

7. If O be *on* the circle ABC, as it will be when P is equal to the supplement of A, the circle (OC) cannot determine any point D by its intersection with ABC, unless P be at P′ in the bisector of BAC, and then the circles coincide altogether. This shows that in this case no quadrilateral of the required species can be constructed with its vertex at any other point than P′; and further that all such quadrilaterals as have their vertex at P′ will then have their sides through P′ equal to each other, as shown in Art. 2. A circle may, in fact, be drawn round *any one* of these quadrilaterals, and the sides through P′ will then be the chords of equal arcs of this circle.

8. Let BO, CO meet the circle ABC in B′, C′, and draw AC′, AB′ cutting BC in M, N; then, for any values of the given angle, when P is at M or N, one of the two points Q, R will be at A; hence no *convex* quadrilateral of the maximum or minimum series can have its vertex *without* MN, which is therefore the greatest linear range of the vertex of such a *convex* quadrilateral on BC.

As the given angle P (or BOC) becomes more nearly equal to the supplement of A, the range MN decreases, and when P is equal to the supplement of A, the points M and N both coincide with P′, which is then in the bisector of A.

When the vertex P is in BM or CN, the maximum or minimum quadrilateral will be *intersecting*, and, in accordance with the usual conventions (see Townsend's *Modern Geometry*, vol. i., art. 118), the part of it which lies *outside* the indefinite space between AB and AC must be considered *negative*. In fact, if a point be supposed to move cyclically round the quadrilateral, always in the same direction AQPRA, the area on the *right* (as we look in the direction of the point's motion) will be *positive*, and that on the *left negative*.

When the angle P is equal to two right angles, QPR becomes a straight line; hence (*a*) includes as a particular case the following theorem, otherwise deduced in Townsend's *Geometry*, art. 48:—

Of all straight lines passing through a fixed point, that which forms with two fixed lines the triangle of *minimum* area, is the one whose segment between the lines is bisected at the point.

When the angle P is greater than two right angles, the quadrilateral AQPR will be *re-entrant*.

9. Considering the *entire series* of quadrilaterals which have a common vertex at a *fixed* point *within* BAC, let us trace the variations in the area of the quadrilateral as the revolving angle turns from its farthest position on the *right* to the farthest on the *left*. If P be *less* than the supplement of A, its farthest position on the right will be when the point R is at infinity, Q at a finite distance along AC, and PR parallel to BA. The quadrilateral will then have an *infinite negative* area; and as the angle P revolves towards the left, the area will *increase* up to zero, will then change its sign and continue to increase till it attains its *maximum* value, which it will do when PQ is equal to PR. As the angle continues to revolve from this *maximum* position, the area of the quadrilateral decreases down to zero, and then passes off to *negative infinity* in the farthest position to the left.

But if P is *greater* than the supplement of A, its farthest position to the right will be when PQ is parallel to AC, and R meets BA at a *finite* distance from A. The quadrilateral has then an *infinite positive* area, which, by the revolution of P to the left, decreases down to its *minimum* value, when PQ is equal to PR, and passes off to *positive infinity* in its farthest position to the left. If we suppose the angle to turn through the remainder of a complete revolution round P, the area of the quadrilateral will, in the *first* case, decrease from positive infinity down to a minimum, and then increase up to positive infinity again; but, in the *second* case, it will increase from negative infinity up to a maximum (*i. e.*, a minimum considered absolutely, without regard to sign), and then decrease to negative infinity; K being in both cases greater than two right angles.

When P is *equal* to the supplement of A, the area of the quadrilateral is *constant* throughout the whole of the revolution if P is in the bisector of A, but otherwise it varies continuously from *positive* to *negative infinity*, and has neither a maximum nor a minimum value.

As the point P moves in either direction from P′ along the indefinite line BC, the quadrilateral determined by the isosceles triangle *increases* or *decreases* without limit according as the angle P is *less* or *greater* than the supplement of A; and when P moves *outside* BAC, the quadrilateral changes in species from a maximum to a minimum, or *vice versâ*. Moreover, the *unique* position (P′) of P, at which the sides make equal angles with BC, will clearly be *within* BAC unless C is an

obtuse angle greater than $\frac{1}{2}(\pi + P)$, when P′ will be to the right of C; and as, for this case to occur, P must be *less* than the supplement of A, this quadrilateral will be the *least* of the series.

10. From Art. 1 we may obtain a solution of the following problem from Swale's *Apollonius*.

To find a point in the base of a triangle such that, if lines be drawn therefrom *parallel to two lines given in position*, they may cut from the triangle a quadrilateral of maximum area.

Here the sides (P′Q′ and P′R′, Fig. 3) of the quadrilateral have not only to contain a given angle, but also to make given angles with the sides of the triangle; θ and ϕ are therefore *constant;* hence, in this case,

$$\frac{du}{d\beta} = 0 \text{ gives } \frac{\beta \sin(\theta + C)}{\sin\theta} = \frac{\gamma \sin(\phi + B)}{\sin\phi},$$

that is, Q′ and R′ must be *equidistant from* BC; also $\dfrac{d^2u}{d\beta^2}$ is positive; hence the quadrilateral (AQ′P′R′) will be a *maximum* when its diagonal (Q′R′) is *parallel to* BC. It may be constructed by drawing (Fig. 3) BO and CO parallel to the lines given in position; then joining AO cutting BC in P′; and through P′ drawing P′Q′, P′R′ parallel to BO, CO. For Q′R′ is then parallel to BC, and P′Q′, P′R′ parallel to the lines given in position.

Mr. Hopps gives the following Solution:—

$\triangle BR'P' : (BP')^2 = \text{a given ratio} = m : 1$ (say);

$\triangle CQ'P' : (P'C)^2 = \text{a given ratio} = n : 1$ (say);

$\therefore\ \triangle BR'P' + CQ'P' = m(BP')^2 + n(P'C)^2,$

and this is to be a *minimum*, which will be the case when P′ is a point in BC such that

$$BP' : P'C = n : m, \text{ or } m(BP') = n(P'C),$$

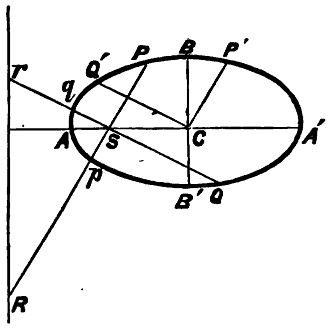
as Mr. Hopps has shown by an ingenious geometrical process in his solution of Quest. 1223, in the *Educational Times* for January, 1862.

It may be readily shown that this result is equivalent to that deduced in the former solution. For, obviously, $2m(\cot P'BR' + \cot BP'R') = 1 = 2n(\cot P'CQ' + \cot CP'Q')$; also $2m(BP')$ and $2n(P'C)$ are the distances of R′ and Q′ from BC.

1427 (Proposed by R. Tucker, M.A.)—Two focal chords (PSp, QSq) of a central conic at right angles to each other are produced to meet the nearer directrix in R, r, and central radii CP′, CQ′, are drawn parallel to them: show that, if BCB′ be the minor axis,

$$\frac{(CP')^2}{RP \cdot Rp} + \frac{(CQ')^2}{rQ \cdot rq} = \frac{(CS)^2}{(CB)^2}.$$

Solution by R. Tucker, M.A.; and Mr. J. Wilson.



Let $2a$, $2b$ be the principal axes, e the eccentricity, and $\angle ASq = \theta$; then we shall have

$$rq = \frac{a(1-e^2)}{e\cos\theta} - \frac{a(1-e^2)}{1+e\cos\theta} = \frac{a(1-e^2)}{e\cos\theta(1+e\cos\theta)}$$

$$\text{also, } rQ = \frac{a(1-e^2)}{e\cos\theta(1-e\cos\theta)};$$

$$\therefore\ rQ \cdot rq = \frac{a^2(1-e^2)^2}{e^2\cos^2\theta(1-e^2\cos^2\theta)} = \frac{b^2(CQ')^2}{a^2e^2\cos^2\theta};$$

$$\text{similarly, } RP \cdot Rp = \frac{b^2(CP')^2}{a^2e^2\sin^2\theta};$$

$$\therefore\ \frac{(CP')^2}{RP \cdot R} + \frac{(CQ')^2}{rQ \cdot rq} = \frac{a^2e^2}{b^2} = \frac{(CS)^2}{(CB)^2}.$$

1463 (Proposed by Dr. Salmon, F.R.S.)

Find the locus of the centre of

(α) A conic which cuts four given finite straight lines harmonically;

(β) A conic which passes through two given points, and cuts two given finite straight lines harmonically;

(γ) A circle which cuts two given finite straight lines harmonically.

Solution by Dr. Hirst, F.R.S.; the Rev. R. Townsend, M.A.; and Mr. G. O. Hanlon.

The geometrical solutions of these problems depend upon the following four simple theorems; the demonstrations of which, for the sake of brevity, are merely indicated.

I. *Through four fixed points only one conic can be drawn so as to cut a given finite line harmonically.*

In fact, the several conics forming the pencil, whose base consists of the four fixed points p, q, r, s, intersect every line A in the plane in a system of points in involution; which system is determined by the intersections of any two conics of the pencil, and most easily by any two of the three pairs of right lines which that pencil in-

cludes. But the several pairs of points, each of which divides harmonically the given finite segment a_1a_2 of A, also determine an involution, of which a_1 and a_2 are the double points; and, as is well known, two collinear involutions have in common only one pair of conjugate points.

II. *All conics which pass through two fixed points, and likewise divide harmonically each of two given finite lines, pass through two other fixed points.*

For p and q being the given fixed points, C any conic fulfilling the conditions in question, and r and s the remaining two intersections of any other pair of conics of the system; every conic of the pencil ($pqrs$) will, like C, divide the given lines harmonically, by (I). But the unique conic of this pencil which passes through either of the intersections of C with the line A, upon which one of the given segments a_1a_2 is situated, must pass through the other; and, doing so, will, by (I), necessarily coincide with C.

III. *Only one conic can be drawn through two fixed points so as to cut three given lines harmonically.*

For, in virtue of its dividing *two* of the given lines harmonically, every such conic would likewise pass through two other fixed points r and s, by (II); and, by (I), only one conic of the pencil ($pqrs$) cuts the third given line harmonically.

IV. *All conics which divide harmonically four given finite lines, likewise pass through four fixed points.*

For C being any such conic, and p, q, r, s the intersections of any two others; every conic of the pencil ($pqrs$) will, like C, divide the four given lines harmonically. Hence the conic of this pencil which passes through either of the intersections of C with one of the given lines will, by (I), pass through the other, and doing so will necessarily coincide altogether with C.

The locus required in (α) is, therefore, identical with that of the centres of the several conics passing through four fixed points, viz. the intersections p, q, r, s of *any* two conics satisfying the required conditions. This locus is well known to be a conic passing through nine easily constructed points, viz. the middle points of the six connectors of the tetrastigm $pqrs$, and the intersections of its three pairs of opposite connectors. (See the Solution of Quest. 1443, in the *Educational Times* for March; also Salmon's *Conics*, 4th ed., ex. 3, p. 143, and ex. 15, p. 288.)

In the problem (β) two of the above points, p and q, are given; and the remaining two, r and s, lie on the easily constructed right line which, together with pq, constitute a conic of the system dividing the given lines a_1a_2 and b_1b_2 harmonically. Another conic of the system, upon which r and s also lie, is that which passes through p, q, the intersection i of the given lines, and the two harmonic conjugates i', i'' of this intersection, with respect to a_1a_2 and b_1b_2.

Finally, the problem (γ) is that special case of (β) for which the fixed points p and q coincide with the circular points at infinity. The right line pq being now at infinity, that which contains r and s bisects each of the segments a_1a_2, b_1b_2, and the conic ($pqii'i''$) becomes a circle. All the circles of the system, therefore, are co-axal, and have rs for radical axis. Consequently the locus of their centres is a right line bisecting rs perpendicularly; it is obviously also the radical axis of the two circles whose diameters are a_1a_2 and b_1b_2.

1466 (Proposed by G. T. SADLER, F.R.A.S.)—A purse contains 14 coins; 4 are half-sovereigns, 4 are half-crowns, and the other six are equal to each other in value. Find what that value must be in order that the expectation of receiving 3 coins at random from the purse may be worth 12 shillings.

Solution by MR. H. HOSKINS; MR. S. BILLS; MR. J. WILSON; MR. H. MURPHY; MR. A. RENSHAW; and the PROPOSER.

Suppose the purse to contain $n_1, n_2 \ldots$ coins, of the respective values $v_1, v_2 \ldots$; then the value of the whole contents of the purse is $n_1v_1 + n_2v_2 + \ldots$ or $\Sigma(nv)$: hence, if e be the value of the expectation of receiving s coins at random from the purse, we must have

$$e = \frac{s}{\Sigma(n)}\Sigma(nv)\ldots\ldots\ldots\ldots\ldots\ldots(\alpha).$$

In the particular case proposed, $n_1 = n_2 = 4$, $n_3 = 6$, $v_1 = 10$, $v_2 = 2\frac{1}{2}$, $e = 12$, $s = 3$; hence

$$\Sigma(nv) = 56, \quad n_3v_3 = 6, \quad v_3 = 1.$$

The 6 coins are therefore *shillings*.

1473 (Proposed by HUGH GODFRAY, M.A.) Solve the equation

$$(x^2-2x)(x^2-4) = 2.$$

Solutions (1) by MR. T. POOLEY; MR. S. BILLS; MATTHEW COLLINS, B.A.; R. TUCKER, M.A.; G. T. SADLER, F.R.A.S.; MR. H. MURPHY; MR. A. RENSHAW; and the PROPOSER; (2) by W. EASTERBY, B.A., St. Asaph Grammar School; and MR. J. WILSON.

1. The given equation becomes, successively,

$$x^4-2x^3-4x^2+8x-2 = 0;$$

$$(x^4-2x^3+x^2)-2(x^2-x)+1 = 3(x-1)^2;$$

$$x^2-x-1 = \pm(x-1)\sqrt{3};$$

whence the four roots are

$$\tfrac{1}{2}\left\{1+\sqrt{3}\pm\sqrt{(8-2\sqrt{3})}\right\},$$

$$\tfrac{1}{2}\left\{1-\sqrt{3}\pm\sqrt{(8+2\sqrt{3})}\right\}.$$

2. Otherwise, put $x = 1-z$; then

$$z^4-2z^3-4z^2+2z+1 = 0;$$

$$(z-z^{-1})^2-2(z-z^{-1})-2 = 0;$$

$$z-z^{-1} = 1\pm\sqrt{3};$$

whence we obtain the same values of x as in the first Solution.

1474 (Proposed by G. T. SADLER, F.R.A.S.) Find x and y from the equations

$$\frac{x^3}{y} = 7250 - xy \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{y^3}{x} = xy - 840 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solution by W. MARTIN, F.R.A.S., Trafalgar, Salisbury; Mr. J. WILSON; Mr. S. BILLS; W. EASTERBY, B.A.; M. COLLINS, B.A.; R. J. NELSON, M.A.; Mr. W. HOPPS; Mr. T. POOLEY; R. TUCKER, M.A.; Mr. H. MURPHY; Mr. A. RENSHAW; and the PROPOSER.

(1) × (2) gives $xy = 1000$, or 3045........(3).

From (1) and (3), $x^4 = 6250000$, or 12804225;

From (2) and (3), $y^4 = 160000$, or 6714225;

$$\therefore \begin{cases} x = 50, \text{ or } \sqrt{(5^2 \cdot 21 \cdot 29^3)}; \\ y = 20, \text{ or } \sqrt[4]{(5^2 \cdot 21^3 \cdot 29)}. \end{cases}$$

1272 (Proposed by the EDITOR.) A heavy uniform rod is thrown at random on a round table; required the respective probabilities of its resting (1) wholly *on* the table, (2) with *one* end over the edge of the table, (3) with *both* ends over the edge of the table.

Solution by the PROPOSER.

Let r be the radius of the table, $2a$ the length of the rod, and x the distance of the *middle* of the rod from the centre of the table; and, when one end of the rod just lies at the edge of the table, let θ, ψ be the angles opposite to the sides r, x of the triangle formed by the lines (r), (a), (x).

At every position of its middle point, suppose the rod to perform an entire revolution through two right angles; then, for any given position of the middle of the rod, the chances of (1), (2), (3) will clearly be the ratios of the angles which favour these cases to two right angles.

Supposing, first, that a is not greater than r, put $h = r - a$, and $k = \sqrt{(r^2 - a^2)}$.

Then, when x is less than h, the rod must rest wholly on the table; when x is between h and k, the rod will rest wholly on the table while it revolves through an angle $(4\theta - 2\pi)$, and throughout the other part $(4\pi - 4\theta)$ of its revolution, *one* of its ends will be over the edge of the table; and when x is between k and r, *one* of its ends will be over the edge of the table while the rod revolves through an angle 4θ, but during the remaining part $(2\pi - 4\theta)$ of its revolution, *both* ends will be over the edge of the table.

Also the probability that the middle of the rod will lie on an annular element, whose bounding radii are $(x, x + \Delta x)$, is $2\pi x \,\Delta x : \pi r^2$, or $2x\,\Delta x : r^2$.

Hence, putting p_1, p_2, p_3 for the respective probabilities of (1), (2), (3), we shall have

$$2\pi r^2 p_1 = \left\{ \int_0^h 2\pi + \int_h^k (4\theta - 2\pi) \right\} 2x\,dx,$$

$$2\pi r^2 p_2 = \left\{ \int_h^k (4\pi - 4\theta) + \int_k^r 4\theta \right\} 2x\,dx,$$

$$2\pi r^2 p_3 = \int_k^r (2\pi - 4\theta)\, 2x\,dx.$$

Now $2x\,dx = d(x^2)$, and we readily find that

$$\int \theta d(x^2) = x^2\theta + r^2\psi - ar \sin\psi;$$

hence, putting $a = r\cos\lambda = 2r\cos\mu$, and denoting the function $(\lambda - \sin\lambda)$ by $F(\lambda)$, the values of p_1, p_2, p_3 will be given by the expressions

$$\pi p_1 = F(2\lambda),$$

$$\pi p_2 = 2F(2\mu) - 2F(2\lambda),$$

$$\pi p_3 = F(\pi) + F(2\lambda) - 2F(2\mu).$$

When the length of the rod is equal to the diameter of the table (or $a = r$), we shall have

$$\lambda = 0, \ \mu = \frac{1}{3}\pi, \ F(2\lambda) = 0, \ F(2\mu) = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3};$$

$$\therefore p_1 = 0, \ p_2 = \frac{4}{3} - \frac{\sqrt{3}}{\pi}, \ p_3 = \frac{\sqrt{3}}{\pi} - \frac{1}{3};$$

or $p_2 : p_3 = 18 : 5$, nearly.

When the length of the rod exceeds the diameter of the table, (or $r < a < 2r$), λ becomes unreal, and $F(2\lambda)$ must be rejected or considered as zero; hence in this case we shall have

$$p_1 = 0, \ p_2 = \frac{2}{\pi}F(2\mu), \ p_3 = 1 - \frac{2}{\pi}F(2\mu).$$

For example, suppose the length of the rod to be equal to the diagonal of a square described about the table; then $\mu = \frac{1}{4}\pi$, $F(2\mu) = \frac{1}{2}\pi - 1$,

$$\therefore p_1 = 0, \ p_2 = 1 - \frac{2}{\pi}, \ p_3 = \frac{2}{\pi};$$

or $p_2 : p_3 = 4 : 7$, nearly.

When the length of the rod exceeds twice the diameter of the table (or $a > 2r$), μ is unreal and $F(2\mu)$ must also be considered as zero; and then $p_1 = 0$, $p_2 = 0$, $p_3 = 1$; that is, the rod must always rest with *both* ends over the edge of the table.

1273 (Proposed by the EDITOR.)—In a given triangle let *three* triangles be inscribed, by joining the points of contact of the inscribed circle, the points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the *corresponding* sides of these three triangles pass through the same point; also the triangle formed by the three points of intersection will be a *circumscribed copolar* to the original triangle, and the *pole* will be on the *straight line* in which the sides of the given triangle meet the bisectors of its exterior angles.

Solution by W. A. Whitworth.

Let the given triangle be taken as triangle of reference. Then the bisector of the angle A meets BC in the point given by

$$\beta = \gamma, \quad \alpha = 0;$$

the perpendicular from A meets it in the point given by

$$\beta \cos B = \gamma \cos C, \quad \alpha = 0;$$

and the point of contact of the inscribed circle is given by

$$\beta\,(1 + \cos B) = \gamma\,(1 + \cos C), \quad \alpha = 0.$$

Hence the corresponding sides c_1, c_2, c_3 of the three inscribed triangles are given respectively by

$$\alpha + \beta - \gamma = 0,$$

$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0,$$

$$\alpha\,(1 + \cos A) + \beta\,(1 + \cos B) - \gamma\,(1 + \cos C) = 0;$$

therefore they all pass through one point, viz.,

$$\frac{\alpha}{\cos B - \cos C} = \frac{\beta}{\cos C - \cos A} = \frac{-\gamma}{\cos A - \cos B} (P_3).$$

Similarly the sides a_1, a_2, a_3 pass through the point

$$\frac{-\alpha}{\cos B - \cos C} = \frac{\beta}{\cos C - \cos A} = \frac{\gamma}{\cos A - \cos B} (P_1),$$

and the sides b_1, b_2, b_3 pass through the point

$$\frac{\alpha}{\cos B - \cos C} = \frac{-\beta}{\cos C - \cos A} = \frac{\gamma}{\cos A - \cos B} (P_2).$$

The equation to the line joining P_1, P_2 is

$$\frac{\alpha}{\cos B - \cos C} + \frac{\beta}{\cos C - \cos A} = 0,$$

and similarly we may write down the equations to P_2P_3, P_3P_1.

Hence the triangle $P_1P_2P_3$ circumscribes the given triangle, and (from the form of the equations) is copolar to it, the pole being the point

$$\frac{\alpha}{\cos B - \cos C} = \frac{\beta}{\cos C - \cos A} = \frac{\gamma}{\cos A - \cos B}$$

which evidently lies on the straight line

$$\alpha + \beta + \gamma = 0,$$

that is, on the straight line in which the sides of the given triangle are cut by the external bisectors of the opposite angles.

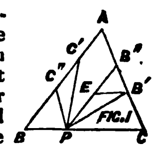
[Note.—According to the nomenclature adopted (from Poncelet and Chasles) in the latest and best works on Geometry (Salmon's *Conics*, 4th ed., pp. 61, 357; Townsend's *Modern Geometry*, vol. i. p. 189) the triangles called in the Question *co-polar* would be, perhaps more appropriately, said to be in *perspective* or *homology*, the point (or pole) in which the lines joining the corresponding vertices meet, and the straight line on which the intersections of the corresponding sides lie, being called respectively the *centre* and *axis* of perspective or homology.—Editor.]

1428 (Proposed by the Editor.)—From a point in the base of a triangle two straight lines are drawn containing a given angle, and forming a quadrilateral with the other two sides of the triangle. It is required to prove that

(α) Of all the quadrilaterals which have a common vertex at the *same* point within the vertical angle of the triangle, that of which the sides passing through this point are equal to each other is a *maximum*, a *minimum*, or *neither* a maximum nor a minimum, according as the given angle is *less* than, *greater* than, or *equal* to, the supplement of the vertical angle of the triangle.

(β) Of all these maximum or minimum quadrilaterals which have their vertices at *different* points in the base, the *least maximum* or the *greatest minimum* is that whose equal sides make equal angles with the base.

Solution by Matthew Collins, B.A.; and R. Tucker, M.A.

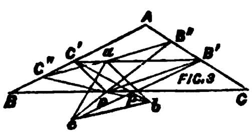


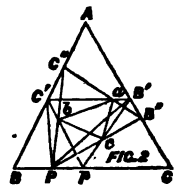
1. Let ABC be the given triangle, P the point in the base BC, B′PC′ a position of the given angle revolving about P such that PB′ = PC′, and B″PC″ another position of the same angle; and suppose, first, that the sum of the two given angles A and P is *less* than two right angles. Then the angles AB′P and AC′P will be greater than two right angles, ∴ ∠ PB′B″ > PC′C″; hence, making ∠ PB′E = PC′C″, we have (Euc. i. 26) △ PC′C″ = PB′E < PB′B″; ∴ AB′PC′ > AB″PC″, that is, the quadrilateral AB′PC′ is a *maximum*.

2. It is clear that a similar reasoning would show that the area of AB′PC′ would be a *minimum* when PB′ = PC′, if A + P > 2 right angles.

3. But if A + P = 2 right angles, it is evident from our demonstration that the triangles PB′B″, PC′C″ would then be equiangular, and would therefore (Euc. vi. 19) be as the squares of their altitudes, which are the distances of P from AC, AB. If therefore P were farther from AC than from AB, the triangle PB′B″ would *always* exceed PC′C″, and the quadrilateral AB′PC′ would increase indefinitely by the revolution of the given angle P towards C: in this case, then, the quadrilateral would admit of *neither* a *maximum* nor a *minimum value*. Moreover, it is evident that when P is in the *bisector* of A, or is equidistant from AB and AC, the triangles PB′B″, PC′C″ are equal to each other in all respects for *any* two positions of the given angle revolving round P; the quadrilateral will then be *constant* in area, and its sides which meet at P will be always equal to each other. We have thus proved the theorems in (α).

4. If the point P moving along BC be so placed at p that the sides (pB′ = pC′) of the given angle (B′pC′) are equally inclined to the base BC, the quadrilateral pB′AC′ will be *minimum maximorum*, that is, the *least* of its *maximum* values, when B′pC′ + A < 2 right angles (Fig. 2); but the same quadrilateral will be *maximum mini-*





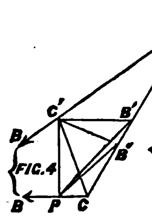
morum, or the greatest of its *minimum* values, when $B'pC' + A > 2$ right angles (Fig. 3). For let $B''PC''$ be another position of the given angle ($B''PC'' = B'pC'$) such that $PB'' = PC''$, and $BP < Bp$; and let the homologous sides of the two similar isosceles triangles $pB'C'$, $PB''C''$, inscribed in ABC, intersect in a, b, c, respectively opposite to A, B, C. Then, since

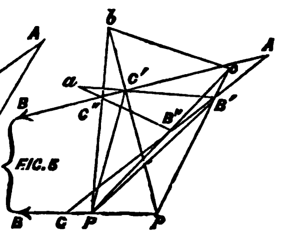
$$\angle bPc = B''PC'' = BpC' = bpc,$$

the quadrilateral $bcpP$ is inscriptible in a circle, and therefore (Euc. iii. 21, 22) the straight line joining *any* two of these four points subtends equal or supplemental angles at the remaining two points; and as, by hypothesis, Pb and Pc subtend supplemental angles at p, these two lines must subtend equal angles at c and b, $\therefore\ Pb = Pc$ (Euc. i. 6). But $PC'' = PB''$, $\therefore\ bC'' = cB''$; and because $\angle Pbc = Pcb$, $\therefore\ pcb > pbc$ in Fig. 2, but $< pbc$ in Fig. 3; $\therefore\ pb > pc$ in Fig. 2, but $< pc$ in Fig. 3; and as $pB' = pC'$, therefore in *both figures* $cB' > bC'$. Again, $\angle B'cB'' = C'bC''$, and $cB'' = bC''$; $\therefore\ \Delta B'cB'' > C'bC''$, and since

$$\Delta PB'B'' : cB'B'' = PB'' : cB'' = PC'' : bC''$$
$$= \Delta PC'C'' : bC'C'',$$

$\therefore\ \Delta PB'B'' > PC'C''$. Let these last unequal triangles be *added* to $PB'AC''$ in Fig. 2, or *subtracted* from it in Fig. 3, and we obtain $PB'AC'$ ($=pB'AC'$, by Euc. i. 37, since $B'C'$ is clearly parallel to BC) $< PB''AC''$ in Fig. 2, but $> PB''AC''$ in Fig. 3. The proof holds good, *mutatis mutandis*, when $BP > Bp$; and thus the theorems in (β) are proved.





5. When two similar triangles $pB'C'$, $PB''C''$, whether isosceles or not, are inscribed in another triangle ABC, so that their homologous sides opposite the angles A, B, C intersect in a, b, c respectively; then any two of these three points, such as b and c, will be *always* on the same side of the line BC corresponding to them; and the area of the triangle ABC, and the point A, will or will not be on the same side of BC as b and c are, according as $\angle BAC + B'pC' <$ or > 2 right angles, and when $\angle BAC + B'pC' = 2$ right angles, the four points $bcpP$ coincide with each other at p upon BC. (See Figs. 2, 3, 4, 5.)

For as $\angle B'pC' = B''PC''$, $\therefore\ \angle B'pP + B''Pp = C''Pp + C'pP$, and according as each of these two equal sums is *less* or *greater* than two right angles, *both* points c and b will be above BC or *both* below BC, by Euclid's 12th axiom. In like manner, a and b are *both* on the same side of AB, and a and c are *both* on the same side of AC. If $\angle BAC + B'pC' = 2$ right angles, then, as $\angle B''PC'' = B'pC' =$ supplement of A, we should have (by Euc. iii. 21, and hyp.) $\angle pAB' = pC'B' = PC''B'' = PAB''$, which is impossible unless P (and thence b and c) coincide with p.

In like manner, a, b, and C'' should coincide with C' (Fig. 4), if $\angle C + pC'B' = 2$ right angles; and similarly for $acB''B'$. Again, the similar triangles $B'pC'$, $B''PC''$ indicate, (by Euc. iii. 21,) that the quadrilaterals $abC'C''$, $bcpP$, $caB'B''$ are inscriptible in circles, $\therefore\ \angle AB'C' + AC'B' = acB'' + abC'' = bPc \pm bac$ (viz., $-$ for Fig. 3, and $+$ for Fig. 2, in which each of the two last mentioned pairs of angles together with $abP + acP$ make 4 right angles); therefore in Fig. 2 (by Euc. i. 32) $A + P + bac = 2$ right angles $=$ (in like manner) $B + pB'C' + abc = C + pC'B' + acb$, so that, in this Fig. 2, $A + B''PC''$, or $B + PB''C''$, or $C + PC''B''$, are each < 2 right angles; hence, when any of the three pairs of given angles $A + P$ or $B + B'$ or $C + C'$ exceeds 2 right angles, Fig. 3 or Fig. 5 will be the proper diagram; and as these three pairs together make 4 right angles (the sum of the angles of the two triangles ABC, $PB''C''$) one pair *only* could exceed 2 right angles; and in Fig. 3 we proved $\angle AB'C' + AC'B' = B''PC'' - bac$, $\therefore\ A + B''PC''$ (Euc. i. 32) exceeds 2 right angles in this figure.

6. Since (Euc. i. 29) 2 right angles $= BpB' + pB'C' = CpC' + pC'B'$, and (Euc. i. 16) *whilst p lies between* B *and* C we have $C < BpB'$ and $B < CpC'$, $\therefore\ B + pB'C'$ or $C + pC'B' < 2$ right angles in Figs. 2 and 3.

By conceiving AC to turn gradually about B', all the other lines in Fig. 2 remaining fixed, it is clear that p can only coincide with C and must do so (Fig. 4) when $\angle C = BpB' = 90° + \frac{1}{2}$ the given $\angle B'pC'$; and then (by Art. 5) C'', a, and b must coincide with C', as $C + pC'B' = BpB' + pB'C' = 2$ right angles; also, in this case $\Delta ACC' = AB'PC' < AB''PC'$. But if $C > BpB'$, i.e. $> 90° + \frac{1}{2}$ the given $\angle P$, we should clearly have $Bp > BC$, and $C + pC'B' > 2$ right angles; hence (by Art. 5) the points a and b will be *outside* the line AB; and it may be readily proved (as in Art. 4) that $\Delta PB'B'' < PC'C''$, and $\therefore$ quadrilateral $AB'pC'$ ($= AB'PC'$) $< AB''PC''$.

1436 (Proposed by MATTHEW COLLINS, B.A.)

If $\tan\theta = \dfrac{y}{x}$, find in terms of y, θ, $\dfrac{dy}{d\theta}$, $\dfrac{d^2y}{d\theta^2}$, what values are to be substituted for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, when the independent variable is changed from x to θ.

Solution by R. TUCKER, M.A.

For shortness' sake, let $\dfrac{dy}{d\theta} = p$, $\dfrac{d^2y}{d\theta^2} = q$;

then $x = y \cot\theta$,

and x, y may be considered functions of θ; hence

$$\frac{dx}{d\theta} = p\cot\theta - y\,\mathrm{cosec}^2\,\theta;$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{p}{p\cot\theta - y\,\mathrm{cosec}^2\,\theta};$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{q}{(p\cot\theta - y\,\mathrm{cosec}^2\,\theta)^2}$$

$$-\frac{p\,(q\cot\theta - 2p\,\mathrm{cosec}^2\,\theta + 2y\,\mathrm{cosec}^2\,\theta\cot\theta)}{(p\cot\theta - y\,\mathrm{cosec}^2\,\theta)^3}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{2p^2 - (p\cot\theta + q)\,y}{\sin^2\theta\,(p\cot\theta - y\,\mathrm{cosec}^2\,\theta)^3}.$$

1461 (Proposed by the EDITOR.)—Show how to transform any given algebraic equation

$$x^n + lx^{n-1} + mx^{n-2} + px^{n-3} + qx^{n-4} + \&c. = 0$$

into another of the form

$$y^n + \mathrm{D}y^{n-4} + \mathrm{E}y^{n-5} + \&c. = 0 \ldots\ldots\ldots\ldots (1),$$

$$\text{or } y^n + \mathrm{C}y^{n-3} + \mathrm{E}y^{n-5} + \&c. = 0 \ldots\ldots\ldots\ldots (2),$$

by the aid of equations in the first case (1) not higher than the third degree, or, in the second case (2), not higher than the fourth degree.

Solution by the Rev. ROBERT HARLEY, F.R.S.

"The transformation by which the second term is taken away from an equation has been long known. In the case of a perfect cubic, it was absolutely necessary in order to the application of the rule of Cardan. This transformation is effected *linearly*, that is to say, the equation connecting the roots of the original and of the transformed equation only contains the first powers of those roots, together with an arbitrary quantity, which is determined so as to satisfy the required condition. This it may be made to do by solving a linear equation. In the same manner, by determining the arbitrary quantity so as to satisfy a certain quadratic, cubic, or biquadratic, the third, fourth, or fifth terms of the transformed equation might be made to vanish. But the linear transformation is inadequate to the removal of more than one term at a time.

"To Tschirnhausen is due the introduction of quadratic and the suggestion of higher transformations. Without this improvement all progress in the theory of transformation would have been stopped. The two different quadratic transformations which he gives for the annihilation of the middle terms of a cubic are well worthy of attention. In the first of them (*Acta* for 1683, p. 206), the equation which connects the roots of the original and transformed equations involved first and second powers of the former roots, and first powers only of the latter; but in the second transformation of a cubic (Ibid. p. 207) the case is reversed, and the original roots only enter to one dimension, while the roots of the transformed equation enter to two." (Mr. Cockle, Analysis of the Theory of Equations, *Philosophical Magazine* for May, 1848.)

If, following Tschirnhausen, we transform the equation

$$x^n + lx^{n-1} + mx^{n-2} + px^{n-3} + qx^{n-4} + \&c. = 0,$$

into another of the form

$$y^n + \mathrm{A}y^{n-1} + \mathrm{B}y^{n-2} + \mathrm{C}y^{n-3} + \mathrm{D}y^{n-4} + \&c. = 0,$$

whose roots are connected with those of the former by the quadratic equation

$$y = \alpha + \beta x + x^2,$$

we shall have

$$\begin{aligned} \Sigma y &= n\alpha + \beta\Sigma x + \Sigma x^2 \\ &= n\alpha - \beta l + l^2 - 2m, \\ \Sigma y^2 &= n\alpha^2 + 2\alpha\beta\Sigma x + (2\alpha + \beta^2)\,\Sigma x^2 + 2\beta\Sigma x^3 + \Sigma x^4 \\ &= n\alpha^2 - 2\alpha\beta l + (2\alpha + \beta^2)\,(l^2 - 2m) \\ &\qquad - 2\beta\,(l^3 - 3lm + p) \\ &\qquad + l^4 - 4l^2m + 4lp + 2m^2 - 4q, \ \&c. \end{aligned}$$

And since, when A and B vanish simultaneously, Σy and Σy^2 must also vanish simultaneously (for $-\mathrm{A} = \Sigma y$, and $-2\mathrm{B} = \mathrm{A}\Sigma y + \Sigma y^2$), it follows that, by the solution of a linear equation and a quadratic equation, we can determine the arbitraries α and β in terms of the known coefficients l, m, p, q, so that the transformed equation in y will be wanting in its second and third terms.

In place, therefore, of the given equation in x, we may work with the reduced form

$$x^n + px^{n-3} + qx^{n-4} + \&c. = 0.$$

Writing $y = a + bx + cx^2 + dx^3 + x^4$,

and transforming as in the Solution of Question 1401 (p. 38) there results an equation of the form

$$y^n + \mathrm{A}y^{n-1} + \mathrm{B}y^{n-2} + \mathrm{C}y^{n-3} + \mathrm{D}y^{n-4} + \&c. = 0,$$

whose coefficients are connected with those of the equation in x by the relations indicated in that solution. We may reduce those relations, however, by observing that, in the present case, $l=0$ and $m=0$, so that $\Sigma x = 0$ and $\Sigma x^2 = 0$; and that, consequently,

$$\begin{aligned} \Sigma y &= na + d\Sigma x^3 + \Sigma x^4, \\ \Sigma y^2 &= na^2 + (2ad + 2bc)\,\Sigma x^3 + (2a + 2bd + c^2)\,\Sigma x^4 \\ &\qquad + (2b + 2cd)\Sigma x^5 + (2c + d^2)\Sigma x^6 + 2d\Sigma x^7 + \Sigma x^8, \\ \Sigma y^3 &= na^3 + (3a^2d + 6abc + b^3)\,\Sigma x^3 + \&c., \end{aligned}$$

(the remaining terms as on p. 38,) and similarly for the higher powers. Now, when A, B, and C vanish simultaneously, so also do Σy, Σy^2, and Σy^3; and when A, B, and D vanish simultaneously, so also do Σy, Σy^2, and Σy^4. (See equations on p. 39, first column.) But $\Sigma y = 0$ gives

$$a = -\frac{1}{n}(d\Sigma x^3 + \Sigma x^4),$$

and substituting this value for a in $\Sigma y^3=0$, we get, after slight reduction,

$$\frac{1}{n}(d\Sigma x^3+\Sigma x^4)^2 = 2b(c\Sigma x^3+d\Sigma x^4+\Sigma x^5)+c^2\Sigma x^4$$
$$+2cd\Sigma x^5+(2c+d^2)\Sigma x^6+2d\Sigma x^7+\Sigma x^8.$$

Make $c\Sigma x^3+d\Sigma x^4+\Sigma x^5=0$,

$$\text{then } \frac{1}{n}(d\Sigma x^3+\Sigma x^4)^2 = c^2\Sigma x^4+2cd\Sigma x^5+(2c+d^2)\Sigma x^6$$
$$+2d\Sigma x^7+\Sigma x^8;$$

and from these two equations we may find, by the solution of a quadratic, the arbitraries c and d in terms of symmetric functions of x. These arbitraries, therefore, and also the arbitrary a, may be considered as known; and if we substitute for them their values in $\Sigma y^3=0$, or $\Sigma y^4=0$, we get on the first alternative a cubic, and on the second a biquadratic in b.

[NOTE.—We have received a Solution of this Question from our valued correspondent Mr. S. Bills, but as it does not differ essentially from the above, we have not thought it necessary to publish it. Mr. Bills observes that when A, B, and C severally vanish, the following relations obtain, viz., $D=-\frac{1}{4}\Sigma y^4$, and $E=-\frac{1}{5}\Sigma y^5$; also that when A, B, and D severally vanish, then $C=-\frac{1}{3}\Sigma y^3$, and $E=-\frac{1}{5}\Sigma y^5$. This indeed is obvious on inspecting the table connecting the coefficients A, B, &c. with the sums of the powers of y on p. 39.—EDITOR.]

1467 (Proposed by HUGH GODFRAY, M.A.) n counters are marked with the numbers $1, 2, 3, \ldots n$ respectively. Show that the number of ways in which three may be drawn, so that the greatest and least together may be double the mean, is

$$\tfrac{1}{4}n(n-2)+\tfrac{1}{8}\{1-(-1)^n\}.$$

Solution by Mr. H. HOSKINS; and ALPHA.

The greatest and least of the counters drawn must be both even or both odd; and for every such pair there will be a third counter equal to half the sum of the other two, so that each of the triads thus obtained will satisfy the conditions of the Question.

If n be even, the number of ways of drawing 2 even numbers will be the combinations of the $\frac{1}{2}n$ even counters taken 2 together, that is $\frac{1}{8}n(n-2)$, which is also the number of ways of drawing 2 odd numbers, since there are as many odd as even numbers. Hence the number of different ways of drawing, when n is even, will be

$$\tfrac{1}{4}n(n-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha).$$

If n be odd, the number of ways of drawing 2 even numbers will be the combinations of the $\frac{1}{2}(n-1)$ even counters taken 2 together, that is, $\frac{1}{8}(n-1)(n-3)$; and of drawing 2 odd numbers, the combinations of the $\frac{1}{2}(n+1)$ odd numbers 2 together, or $\frac{1}{8}(n-1)(n+1)$. Hence the number of different ways of drawing, when n is odd, will be

$$\tfrac{1}{4}(n-1)^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

Both results are included in the expression

$$\tfrac{1}{4}n(n-2)+\tfrac{1}{8}\{1-(-1)^n\}\ldots\ldots\ldots\ldots(\gamma).$$

[NOTE.—The total number of different triads that can be drawn from n counters is $\frac{1}{6}n(n-1)(n-2)$, and these are all equally likely to appear; hence the *probability* that, if three counters be drawn *at random*, the number on one of them will be half the sum of the numbers on the other two, is

$$\frac{3}{2(n-1)}, \text{ or } \frac{3(n-1)}{2n(n-2)},$$

according as n is even or odd.—EDITOR.]

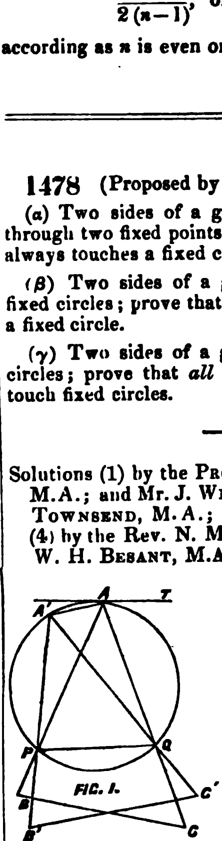
1478 (Proposed by J. MCDOWELL, M.A.)

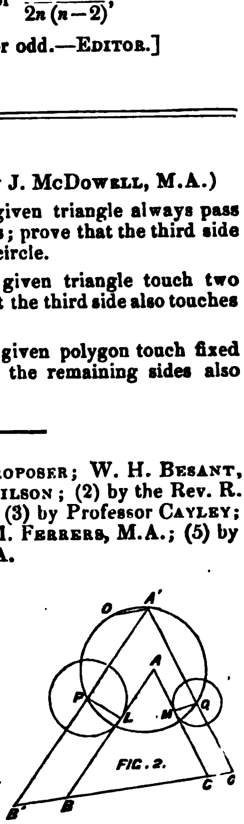
(α) Two sides of a given triangle always pass through two fixed points; prove that the third side always touches a fixed circle.

(β) Two sides of a given triangle touch two fixed circles; prove that the third side also touches a fixed circle.

(γ) Two sides of a given polygon touch fixed circles; prove that *all* the remaining sides also touch fixed circles.

Solutions (1) by the PROPOSER; W. H. BESANT, M.A.; and Mr. J. WILSON; (2) by the Rev. R. TOWNSEND, M.A.; (3) by Professor CAYLEY; (4) by the Rev. N. M. FERRERS, M.A.; (5) by W. H. BESANT, M.A.





1. (α.) Through the fixed points P, Q (Fig. 1) draw a circle containing the given angle opposite the third side. Let A′B′C′ be any given triangle of the system, and find the point A in the circle, so that PA : AQ = C′A′ : A′B′. Produce AP until AB = A′B′, and AQ until AC = A′C′. Draw the tangent TA at A. Because PA : AQ = CA : AB, therefore ∠ ACB = APQ = QAT, and therefore BC is parallel to TA. Join AA′; then ∠AA′Q = APQ = A′C′B′, therefore AA′ and B′C′ are parallel.

Therefore the perpendicular from A on B′C′ is equal to the perpendicular from A′ on B′G,

that is, to the constant perpendicular on the third side of the triangle. Hence the third side always touches the circle, with centre at the fixed point A, and radius equal to the perpendicular on the third side of the triangle from the opposite angle.

(β.) Let P and Q (Fig. 2) be the centres of the two given circles, and ABC any triangle of the system. Through P draw A′B′ parallel to AB, and through Q draw A′C′ parallel to AC, thus forming the triangle A′B′C′. Now the triangle A′B′C′ is clearly given, and two of its sides pass through the fixed points P and Q; therefore, by (α), B′C′ touches a fixed circle.

(γ.) If any *two* sides of a given polygon touch fixed circles, *all* the remaining sides will touch fixed circles: for any other side forms a given triangle with the two given sides.

2. The theorems may be otherwise proved by taking (β) first. Thus (Fig. 2) draw A′O parallel to the third side BC of the given triangle ABC; then it is clear that the point A′ is fixed *relatively* to the triangle ABC, and moves with it on a fixed circle passing through P and Q, the centres of the given circles. The distance of A′ or O from BC is therefore given; and, since the angle OA′P is equal to the given angle B′ or B, A′O passes through a fixed point O on the circle OPQ which is the locus of A′; therefore BC touches the circle whose centre is O, and radius equal to the given distance of A′ from BC.

The theorem (β) includes (α), and (γ) follows from it, as in the foregoing Solution.

3. Since the theorem (γ) follows at once from (β), and (α) is included in (β), it is only necessary to prove (β). Consider three given circles, and let it be proposed to construct a triangle the sides whereof touch the given circles, and which is similar to a given triangle: the direction of one side may be assumed at pleasure, and then the triangle is determined. Impose now on the triangle the condition that the area is equal to a given quantity; we obtain for the given area an expression involving the angle θ which fixes the direction of one of the sides, and we have thus an equation for the determination of the angle θ. But, for a properly determined relation between the data of the problem, the expression for the area becomes independent of the angle θ, that is, every triangle, the sides whereof touch the three circles, and which is similar to a given triangle, is of the same area, or say, the area of every such triangle is equal to a given quantity Δ; and this being so, it is clear, that if we construct a triangle similar to a given triangle and of the given area Δ (that is, a triangle *equal* to a given triangle), in such manner that two of the sides touch two of the given circles, then the envelope of the remaining side will be the remaining given circle; which is in fact the theorem (β).

It only remains therefore to show that the foregoing porismatic case of the problem exists.

For the first circle, let the coordinates of the centre be a, b, and the radius be c; and suppose in like manner that we have a', b', and c' for the second circle, and a'', b'', and c'' for the third circle. Let λ, λ', λ'' be the inclinations to the axis of x of the perpendiculars on the sides which touch these circles respectively; then the equations of the three sides respectively are

$$(x-a)\cos\lambda+(y-b)\sin\lambda-c=0,$$
$$(x-a')\cos\lambda'+(y-b')\sin\lambda'-c'=0,$$
$$(x-a'')\cos\lambda''+(y-b'')\sin\lambda''-c''=0.$$

And if the triangle be similar to a given triangle, then the differences of the angles λ, λ', λ'' will be given angles, or what is the same thing, we may write

$$\lambda=\theta+\xi,\quad \lambda'=\theta'+\xi,\quad \lambda''=\theta''+\xi,$$

where θ, θ', θ'' are given angles, and ξ is a variable angle. Let Δ be the area of the triangle, then (disregarding a merely numerical factor) we have

$$\begin{aligned}\sqrt{\Delta}= &\sin(\lambda'-\lambda'')(a\cos\lambda+b\sin\lambda+c)\\ &+\sin(\lambda''-\lambda)(a'\cos\lambda'+b'\sin\lambda'+c')\\ &+\sin(\lambda-\lambda')(a''\cos\lambda''+b''\sin\lambda''+c'');\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}\sqrt{\Delta}= &\\ &\sin(\theta'-\theta'')\left\{a\cos(\theta+\xi)+b\sin(\theta+\xi)+c\right\}\\ &+\sin(\theta''-\theta)\left\{a'\cos(\theta'+\xi)+b'\sin(\theta'+\xi)+c'\right\}\\ &+\sin(\theta-\theta')\left\{a''\cos(\theta''+\xi)+b''\sin(\theta''+\xi)+c''\right\}.\end{aligned}$$

And it is clear that the right-hand side will be independent of ξ, if only

$$\begin{aligned}&\sin(\theta'-\theta'')(a\cos\theta+b\sin\theta)\\ &+\sin(\theta''-\theta)(a'\cos\theta'+b'\sin\theta')\\ &+\sin(\theta-\theta')(a''\cos\theta''+b''\sin\theta'')=0,\end{aligned}$$

and

$$\begin{aligned}&\sin(\theta'-\theta'')(-a\sin\theta+b\cos\theta)\\ &+\sin(\theta''-\theta)(-a'\sin\theta'+b'\cos\theta')\\ &+\sin(\theta-\theta')(-a''\sin\theta''+b''\cos\theta'')=0;\end{aligned}$$

equations which show that, given the form of the triangle and the centres of two of the circles, the centre of the third circle (in the porismatic case) is a determinate unique point. And the theorem is thus proved.

4. Another analytical proof of (β) may be obtained as follows. Let α, β, γ be the perpendiculars from a fixed point on the sides a, b, c of the triangle, and ϕ, ϕ', ϕ'' the angles which these perpendiculars make with a fixed line: then

$$a\alpha+b\beta+c\gamma=a\ constant;$$

also α, β, γ are functions of ϕ, ϕ', ϕ'', which differ by constants,

$$\therefore\ a\left(\alpha+\frac{d^2\alpha}{d\phi^2}\right)+b\left(\beta+\frac{d^2\beta}{d\phi'^2}\right)+c\left(\gamma+\frac{d^2\gamma}{d\phi''^2}\right)$$
$$=a\ constant,$$

hence, if ρ, ρ', ρ'' be the radii of curvature of the envelopes of a, b, c, we shall have

$$a\rho+b\rho'+c\rho''=a\ constant;$$

and therefore if ρ and ρ' be constant, ρ'' is also constant; that is, if a and b touch two fixed circles, c will also touch a fixed circle.

5. We give another geometrical and also an analytical proof of (a). Let ABC be the moving triangle (Fig. 3), P and Q the fixed points. Draw AO parallel to BC, and let it meet the circle in which A moves in O; then $\angle$ OAC = ACB = a *given* angle, therefore O is a fixed point: also the perpendicular (OT) from O on BC is equal to the (*constant*) perpendicular (AD) from A on BC; hence BC touches a fixed circle whose centre is O, and radius OT or AD.

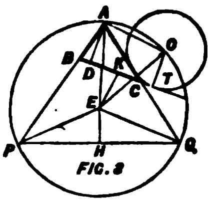


FIG. 3

Analytically.—From the centre (E) of the circle OPQ draw EH perpendicular to PQ, and EK perpendicular to BC; and let R be the radius of OPQ, and ϕ the angle which EK makes with HE produced.

Then $\angle \mathrm{AEK} = \mathrm{EAD} = \mathrm{EAP} - \mathrm{DAB}$

$$= (90^\circ - \mathrm{AQP}) - (90^\circ - \mathrm{ABC}) = \mathrm{B} - \mathrm{Q}.$$

and $\angle \mathrm{Q} = (\phi + 90^\circ) - (90^\circ - \mathrm{C}) = \mathrm{C} + \phi;$

$$\therefore\ \angle \mathrm{AEK} = \mathrm{B} - \mathrm{C} - \phi.$$

Hence, taking axes through E parallel and perpendicular to PQ, the equation of BC is

$$x \sin\phi + y\cos\phi = \mathrm{R}\cos(\mathrm{B}-\mathrm{C}-\phi) - c\sin\mathrm{B},$$

and this can be written in the form

$$(x-\alpha)\sin\phi + (y-\beta)\cos\phi = -\gamma,$$

which is evidently a tangent to the circle

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2.$$

Moreover $\alpha = \mathrm{R}\sin(\mathrm{B}-\mathrm{C})$, and $\beta = \mathrm{R}\cos(\mathrm{B}-\mathrm{C})$,

$$\therefore\ \alpha^2+\beta^2 = \mathrm{R}^2, \text{ and } \frac{\beta}{\alpha} = \cot(\mathrm{B}-\mathrm{C});$$

whence it appears that the centre (O) is on the circumference of the circle APQ, and the angular distance of O from Q is

$$\mathrm{OEQ} = 90^\circ - (\mathrm{B}-\mathrm{C}) + \mathrm{EQP} = 2\mathrm{C},$$

in accordance with the geometrical Solution.

Finally, $\gamma = c\sin\mathrm{B}$, the altitude of the triangle.

[NOTE.—The theorem (a) is due to the Rev A. W. W. Steele, M.A., Fellow of Caius College, Cambridge.

Our junior readers should bear in mind, in Professor Cayley's Solution, that the double area of the triangle formed by the three lines

$$u_1 \equiv a_1x + b_1y + c_1 = 0,\ u_2 = 0,\ u_3 = 0, \text{ is}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}, \text{ or}$$

$$\{c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)\}^2 \div$$
$$[(a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)],$$

which, if the equations are of the form
$u_1 \equiv x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0,\ u_2 = 0,\ u_3 = 0,$ becomes

$$\{p_1\sin(\alpha_2-\alpha_3) + p_2\sin(\alpha_3-\alpha_1) + p_3\sin(\alpha_1-\alpha_2)\}^2 \div$$
$$[\sin(\alpha_1-\alpha_2)\sin(\alpha_2-\alpha_3)\sin(\alpha_3-\alpha_1)],$$

(see Salmon's *Conics*, 4th ed., art. 39); moreover, in Mr. Ferrers's Solution, if (p, ϕ) be the *tangential polar coordinates* of a point in a curve (*i. e.*, the perpendicular p from a fixed point on the tangent and the inclination ϕ of that perpendicular to a fixed straight line), and ρ the radius of curvature at that point, then

$$\rho = p + \frac{d^2p}{d\phi^2},$$

(see Todhunter's *Integral Calculus*, art. 91, 2nd ed.; or Ferrers on the Tangential Polar Equation, *Quarterly Math. Journal*, vol. i.)—EDITOR.]

ELLIPSE AND HYPERBOLA: FAIR EXCHANGE NO ROBBERY.

By Professor DE MORGAN.

IT is well known that the circle on the major axis of an ellipse *or hyperbola* is the locus of the intersection of a tangent to *either curve* with the perpendicular from a focus of the *same curve*. Any one who knows this property of the ellipse might expect beforehand that the same relation would exist between any hyperbola and its equilateral hyperbola which connects any ellipse with its *equilateral ellipse*, or circle. Many years ago, feeling satisfied that if the circle be thus allowed to rob the equilateral hyperbola, the second would somehow or other make reprisals, I found the following theorem, which I gave without demonstration in the *Philosophical Magazine* for 1848, (p. 546.) The following simple demonstration may be interesting, as the theorem has not attracted notice, though the point it clears up is worthy of it. By *supplemental* lines I mean those which make supplemental angles with the positive side of the axis of x.

THEOREM. If an ellipse and an hyperbola have the same major and minor axes, and if from the focus of *either* be drawn a line *supplemental* to the perpendicular on the tangent of *the other*, the intersection of the two lines will lie in the equilateral hyperbola which has the common major axis of the two.

$$\text{Let } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ \frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1$$

be the equation of the ellipse and its tangent at (x, y). The equation of the line supplemental to the perpendicular on the tangent, and drawn through the focus of the hyperbola, is

$$\frac{y\xi}{b^2} + \frac{x\eta}{a^2} = \frac{y}{b^2}\sqrt{(a^2+b^2)}.$$

The second and third equations obviously give

$$\left(\frac{x}{a^2}+\frac{y}{b^2}\right)(\xi+\eta) = 1 + \frac{y}{b^2}\sqrt{(a^2+b^2)},$$

$$\left(\frac{x}{a^2}-\frac{y}{b^2}\right)(\xi-\eta) = 1 - \frac{y}{b^2}\sqrt{(a^2+b^2)}.$$

Multiply these together, for

$$\frac{x^2}{a^4} \text{ write } \frac{1}{a^2}\left(1-\frac{y^2}{b^2}\right),$$

and there results

$$\xi^2-\eta^2=a^2;$$

showing that the straight lines of the second and third equations meet on the equilateral hyperbola. Change the sign of b^2 throughout, and the tangent is drawn to the hyperbola, and the supplemental line passes through the focus of the ellipse.

The young student may look out for the corresponding proof of the well-known robbery committed by the ellipse upon the hyperbola. Those of a higher class will perhaps find other cases.

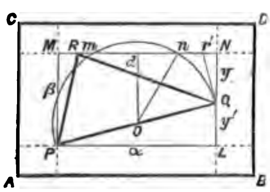
Here is another case. If equal ellipses be placed vertex to vertex, and if one ellipse then roll upon the other, each rolling focus describes a circle about a fixed focus. And if the hyperbola be substituted for the ellipse, the rolling focus still describes a *circle*. What reprisal does the equilateral hyperbola make? I never looked to see, and cannot guess: I leave the matter to the correspondents of the *Educational Times*.

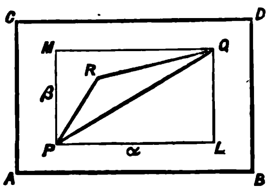
1361 (Proposed by W. S. B. Woolhouse, F.R.A.S.)—If three marbles are thrown at random on the floor of a rectangular room, what is the chance that the triangle which unites them will be acute-angled?

Solution by the Proposer.

The problem may be enunciated geometrically as follows:—

Three points being taken at random on the surface of a given rectangle, determine the probability that the triangle which unites them shall be acute.





Let $AB=a$, $AC=b$ be the sides of the given rectangle ABCD, and suppose P, Q, R to be three points taken at random upon its surface. Conceive the least circumscribing rectangle LM to be drawn having its sides parallel to those of the given rectangle, and just including upon it the triangle PQR. The figure will then be analogous to one or other of the two annexed diagrams. In the first diagram, one of the points is on one of the four corners of the rectangle, the other two points being respectively situated upon the two remote sides, and in this case the triangle may be either acute or obtuse. In the second diagram, two of the points occupy a pair of opposite corners of the rectangle, the third point is anywhere upon its surface, and the triangle is necessarily obtuse.

Proceeding with the first diagram, let $PL=\alpha$, $PM=\beta$, $NQ=y$, $QL=y'$. On PQ as diameter suppose a semicircle to be drawn, cutting MN in m and n, and draw Qr perpendicular to PQ.

The angle P being contained within LPM is necessarily acute; the angle Q will obviously be acute when R is situated between M and r; but in order that the angle R may also be acute, the portion mn must evidently be excluded. Therefore, in respect of the rectangle LM, the range of the point R on the side MN will, for an acute triangle, be restricted to $Mr-mn$.

The triangles PLQ, QNr being similar,

$$PL \, . \, rN = NQ \, . \, QL = yy';$$

$$\therefore\ rN = \frac{yy'}{\alpha}, \text{ and } Mr = \alpha - \frac{yy'}{\alpha}; \text{ also}$$

$$mn^2 = (2On)^2-(2Od)^2 = PQ^2-(PM+QN)^2$$
$$= \alpha^2+(\beta-y)^2-(\beta+y)^2 = \alpha^2-4\beta y.$$

Therefore the linear range of the point R on MN is

$$R = Mr-mn = \left(\alpha-\frac{yy'}{\alpha}\right)-\sqrt{(\alpha^2-4\beta y)};$$

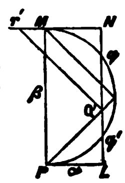
and it must be hereafter remembered that the final term of this expression, or that which represents the value of the segment mn, has a geometrical existence only when the circle cuts the side MN, that is, when the term itself is algebraically real.

Measuring the number of points on a line by its length, the corresponding number of acute triangles having Q, R on LN, MN will be

$$\int R\,dy = \int dy\left(\alpha-\frac{\beta y-y^2}{\alpha}\right)-\int dy\sqrt{(\alpha^2-4\beta y)}$$
$$= \left(\alpha y-\frac{\beta}{2\alpha}y^2+\frac{y^3}{3\alpha}\right)+\frac{(\alpha^2-4\beta y)^{\frac{3}{2}}}{6\beta}$$

(between the limits $y=0\ldots\beta$)

$$= \alpha\beta-\frac{\beta^3}{6\alpha}-\frac{\alpha^3}{6\beta}+\frac{(\alpha^2-4\beta^2)^{\frac{3}{2}}}{6\beta}.$$



But whenever β exceeds 2α, the value of Mr will, between certain limits, become algebraically negative, and therefore situated without the rectangle, as shown in the annexed diagram, and, on that account, the portion of the integral belonging to it should be excluded. The limits are found by making $Mr=0$, and are

$$y = \begin{cases} \frac{1}{2}\beta-\frac{1}{2}\sqrt{(\beta^2-4\alpha^2)} = Nq, \\ \frac{1}{2}\beta+\frac{1}{2}\sqrt{(\beta^2-4\alpha^2)} = Nq'. \end{cases}$$

The integral with respect to Mr, viz., $\alpha y-\dfrac{\beta}{2\alpha}y^2+\dfrac{y^3}{3\alpha}$, estimated between these limits, is $-\dfrac{(\beta^2-4\alpha^2)^{\frac{3}{2}}}{6\alpha}$. Hence, eliminating this, by ap-

plying it with a contrary algebraical sign, we obtain, for the true number of acute triangles having Q, R on LN, MN,

$$N = \alpha\beta - \frac{\alpha^3-(\alpha^2-4\beta^2)^{\frac{3}{2}}}{6\beta} - \frac{\beta^3-(\beta^2-4\alpha^2)^{\frac{3}{2}}}{6\alpha} \,..\, (a).$$

Now the point P may be severally situated on each of the four corners of the bounding rectangle LM. Also the number of admissible positions of this rectangle over the surface of the given rectangle ABCD is obviously represented by $(a-\alpha)(b-\beta)$. Therefore the number of acute triangles that may be drawn on the surface of the given rectangle must be found from the double integral

$$\iint d\alpha\, d\beta\, (a-\alpha)\, (b-\beta) \,.\, 4N \,..........\, (b).$$

Again, the total number of triangles PQR with Q, R on LN, MN, in the first diagram, is evidently $\alpha\beta$, and this becomes quadrupled when P is taken upon each of the four corners of the rectangle. Also the number of triangles PQR, in the second diagram, having P, Q stationed on opposite corners and R anywhere on the surface of the circumscribing rectangle, is $\alpha\beta$, and this becomes doubled when P, Q also occupy the other two opposite corners L, M. The total number, including both diagrams, is therefore $6\alpha\beta$; and when extended throughout the surface of the given rectangle, it gives

$$\iint d\alpha\, d\beta\, (a-\alpha)\,(b-\beta)\,.\,6\alpha\beta = a^3\int \beta\, d\beta\, (b-\beta)$$

$$= \frac{a^3b^3}{2\,.\,3} \,........\, (c).$$

Whence, dividing (b) by (c), the required probability in favour of an acute triangle is expressed by the formula

$$p = \frac{2\,.\,3\,.\,4}{a^3b^3}\iint d\alpha\, d\beta\, (a-\alpha)\,(b-\beta)\,.\,N \,....\, (d).$$

Here the sides α, β of the variable rectangle LM are required to pass through all values less than a, b; and a slight consideration of the geometrical limitations in connexion with the process by which the formula (a) is obtained will show that the terms which exhibit the radical quantities $(\alpha^2-4\beta^2)^{\frac{3}{2}}$, $(\beta^2-4\alpha^2)^{\frac{3}{2}}$ should always be included whenever they are *real*, and excluded, or regarded as zeros, whenever they become *imaginary*. It will follow also that the same rule of exclusion will be applicable to the final integral, when properly stated, since the real and imaginary portions of the result after integration must evidently owe their respective existences to the real and imaginary phases passed through by the several terms of the original formula. In putting down the integrals, we should therefore, as a matter of convenience, exclude all limits between which quantities are necessarily unreal, and at the same time so contrive that the radical quantities and the terms which involve them shall vanish simultaneously, since then it will only be requisite to obliterate these terms whenever they finally take an imaginary form.

Introducing into the formula (d) the terms of the value of N by (a) in due order, the several integrals are obtained as follows:—

$$\iint d\alpha\, d\beta\, (a-\alpha)\, (b-\beta)\,.\,\alpha\beta, \text{ by } (c), = \frac{a^3b^3}{36} \,..\, (e).$$

$$\int d\alpha\, (a-\alpha)\left\{\alpha^3-(\alpha^2-4\beta^2)^{\frac{3}{2}}\right\} = a\frac{\alpha^4}{4} - \frac{\alpha^5}{5}$$

$$- a\left\{\frac{\alpha^2-10\beta^2}{4}\,\alpha\sqrt{(\alpha^2-4\beta^2)}\right.$$

$$\left. + 6\beta^4\log\frac{\alpha+\sqrt{(\alpha^2-4\beta^2)}}{2\beta}\right\} + \frac{(\alpha^2-4\beta^2)^{\frac{5}{2}}}{5}$$

$$\left(\begin{array}{l}\text{the first two terms, } \alpha=0\,....\,a\\ \text{the remaining terms, } \alpha=2\beta\,....\,a\end{array}\right)$$

$$= a^4\frac{a-\sqrt{(a^2-4\beta^2)}}{4\,.\,5} + \beta^2\frac{9a^2+32\beta^2}{10}\sqrt{(a^2-4\beta^2)}$$

$$-6a\beta^4\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta};$$

$$\int\frac{d\beta}{\beta}\left\{a^4\frac{a-\sqrt{(a^2-4\beta^2)}}{20} + \beta^2\frac{9a^2+32\beta^2}{10}\sqrt{(a^2-4\beta^2)}\right.$$

$$\left. -6a\beta^4\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta}\right\}$$

$$= \frac{a^4}{20}\left\{a\log\frac{\beta}{a} + a\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta}\right.$$

$$\left. -\sqrt{(a^2-4\beta^2)}\right\} - \frac{61a^2+96\beta^2}{600}(a^2-4\beta^2)^{\frac{3}{2}}$$

$$- \frac{3a}{2}\beta^4\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta}$$

$$+ a^2\frac{a^2+2\beta^2}{16}\sqrt{(a^2-4\beta^2)}$$

(between $\beta=0\,....\,b$)

$$= \frac{a^5}{20}\log\frac{b}{a} + a\left(\frac{a^4}{20} - \frac{3}{2}b^4\right)\log\frac{a+\sqrt{(a^2-4b^2)}}{2b}$$

$$- \frac{107a^4-446a^2b^2-768b^4}{1200}\sqrt{(a^2-4b^2)}$$

$$+ \frac{107a^5}{1200} \,..............\, (f).$$

$$\int d\beta\left\{a^4\frac{a-\sqrt{(a^2-4\beta^2)}}{20} + \beta^2\frac{9a^2+32\beta^2}{10}\sqrt{(a^2-4\beta^2)}\right.$$

$$\left. -6a\beta^4\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta}\right\}$$

$$= \frac{a^5\beta}{20} - \frac{a^4}{20}\left\{\frac{\beta}{2}\sqrt{(a^2-4\beta^2)} + \frac{a^2}{4}\sin^{-1}\frac{2\beta}{a}\right\}$$

$$+ \frac{9a^2}{10}\left\{-\frac{a^2-8\beta^2}{32}\beta\sqrt{(a^2-4\beta^2)} + \frac{a^4}{64}\sin^{-1}\frac{2\beta}{a}\right\}$$

$$+ \frac{32}{10}\left\{-\left(\frac{a^4}{256} + \frac{a^2\beta^2}{96} - \frac{\beta^4}{6}\right)\beta\sqrt{(a^2-4\beta^2)}\right.$$

$$\left. + \frac{a^6}{512}\sin^{-1}\frac{2\beta}{a}\right\} - \frac{6a\beta^5}{5}\log\frac{a+\sqrt{(a^2-4\beta^2)}}{2\beta}$$

$$+ \frac{6a^2}{5}\left\{\frac{3a^2+8\beta^2}{128}\beta\sqrt{(a^2-4\beta^2)} + \frac{3a^4}{256}\sin^{-1}\frac{2\beta}{a}\right\}$$

(between $\beta=0\,....\,b$)

$$= \frac{a^5 b}{20} - \frac{6ab^5}{5} \log \frac{a+\sqrt{(a^2-4b^2)}}{2b} - \frac{a^6}{160} \sin^{-1} \frac{2b}{a}$$

$$- \left(\frac{3a^4}{80} - \frac{8a^2b^2}{30} - \frac{16b^4}{30}\right) b\sqrt{(a^2-4b^2)} \ldots\ldots (g).$$

Hence $\iint d\alpha\, d\beta\, (a-\alpha)(b-\beta)\, \dfrac{\alpha^3-(\alpha^2-4\beta^2)^{\frac{3}{2}}}{\beta}$

$$= b(f) - (g)$$

$$= \frac{a^5 b}{20} \log \frac{b}{a} + \frac{47a^5 b}{1200}$$

$$+ \frac{a^5 b - 6ab^5}{20} \log \frac{a+\sqrt{(a^2-4b^2)}}{2b}$$

$$+ \frac{a^6}{160}\left(\frac{\pi}{2} - \cos^{-1}\frac{2b}{a}\right)$$

$$- \frac{62a^4 - 126a^2b^2 - 128b^4}{1200} b\sqrt{(a^2-4b^2)} \ldots\ldots (h).$$

Similarly, $\iint d\alpha\, d\beta\, (a-\alpha)(b-\beta)\, \dfrac{\beta^3-(\beta^2-4\alpha^2)^{\frac{3}{2}}}{\alpha}$

$$= \frac{ab^5}{20} \log \frac{a}{b} + \frac{47ab^5}{1200}$$

$$+ \frac{ab^5 - 6a^5 b}{20} \log \frac{b+\sqrt{(b^2-4a^2)}}{2a}$$

$$+ \frac{b^6}{160}\left(\frac{\pi}{2} - \cos^{-1}\frac{2a}{b}\right)$$

$$- \frac{62b^4 - 126a^2b^2 - 128a^4}{1200} a\sqrt{(b^2-4a^2)} \ldots\ldots (k).$$

Therefore, from (a) and (d), the probability of a triangle being *acute* is

$$p = \frac{1}{a^3b^3}\left\{24(e) - 4(h) - 4(k)\right\}$$

$$= \frac{2}{3} - \frac{47}{300}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) + \frac{1}{5}\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\log\frac{a}{b}$$

$$- \left(\frac{a^3}{b^3} + \frac{b^3}{a^3}\right)\frac{\pi}{80}$$

$$+ \frac{1}{40}\left(\frac{a^3}{b^3}\cos^{-1}\frac{2b}{a} + \frac{b^3}{a^3}\cos^{-1}\frac{2a}{b}\right)$$

$$+ \left(31\frac{a^2}{b^2} - 63 - 64\frac{b^2}{a^2}\right)\frac{\sqrt{(a^2-4b^2)}}{150a}$$

$$+ \left(31\frac{b^2}{a^2} - 63 - 64\frac{a^2}{b^2}\right)\frac{\sqrt{(b^2-4a^2)}}{150b}$$

$$- \frac{1}{5}\left(\frac{a^2}{b^2} - \frac{6b^2}{a^2}\right)\log\frac{a+\sqrt{(a^2-4b^2)}}{2b}$$

$$- \frac{1}{5}\left(\frac{b^2}{a^2} - \frac{6a^2}{b^2}\right)\log\frac{b+\sqrt{(b^2-4a^2)}}{2a}.$$

That is, denoting the ratio $\dfrac{b}{a}$ by k,

$$p = \frac{2}{3} - \frac{47}{300}\left(k^2 + \frac{1}{k^2}\right) + \frac{1}{5}\left(k^2 - \frac{1}{k^2}\right)\log k$$

$$- \left(k^3 + \frac{1}{k^3}\right)\frac{\pi}{80} + \frac{1}{40}\left(k^3\cos^{-1}\frac{2}{k} + \frac{1}{k^3}\cos^{-1}2k\right)$$

$$+ \frac{31k^2 - 63 - \frac{64}{k^2}}{150}\sqrt{\left(1-\frac{4}{k^2}\right)}$$

$$+ \frac{\frac{31}{k^2} - 63 - 64k^2}{150}\sqrt{(1-4k^2)}$$

$$- \frac{1}{5}\left(k^2 - \frac{6}{k^2}\right)\log\frac{k+\sqrt{(k^2-4)}}{2}$$

$$- \frac{1}{5}\left(\frac{1}{k^2} - 6k^2\right)\log\frac{\frac{1}{k}+\sqrt{\left(\frac{1}{k^2}-4\right)}}{2} \ldots\ldots (l).$$

The *real* terms of (l) present a complete and general expression for determining the required probability.

If a denote the greater side of the rectangle then k is less than 1, and

$$p = \frac{2}{3} - \frac{47}{300}\left(k^2 + \frac{1}{k^2}\right) + \frac{1}{5}\left(\frac{1}{k^2} - k^2\right)\log\frac{1}{k}$$

$$- \left(k^3 + \frac{1}{k^3}\right)\frac{\pi}{80} + \frac{\cos^{-1}2k}{40k^3}$$

$$+ \frac{\frac{31}{k^2} - 63 - 64k^2}{150}\sqrt{(1-4k^2)}$$

$$- \frac{1}{5}\left(\frac{1}{k^2} - 6k^2\right)\log\frac{1+\sqrt{(1-4k^2)}}{2k} \ldots\ldots (m),$$

the last three terms of which vanish at the point where they become imaginary; and in case the value of k should exceed $\frac{1}{2}$, these terms have simply to be omitted.*

Hence, when k has any value not less than $\frac{1}{2}$ and not exceeding 2, the probability of an acute triangle is accurately determined by the less complicated expression

$$p = \frac{2}{3} - \frac{47}{300}\left(k^2 + \frac{1}{k^2}\right) + \frac{1}{5}\left(\frac{1}{k^2} - k^2\right)\log\frac{1}{k}$$

$$- \left(k^3 + \frac{1}{k^3}\right)\frac{\pi}{80} \ldots\ldots\ldots (n).$$

For a square, $k = 1$, and

$$p = \frac{53}{150} - \frac{\pi}{40} = 0{\cdot}27479.$$

For a circle, it has been found that

$$p = \frac{4}{\pi^2} - \frac{1}{8} = 0{\cdot}28029.$$

(See Solution of Quest. 1350 in the *Educational Times* for October 1863.)

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When k is a fraction less than $\frac{1}{2}$, the numerical calculation will be somewhat simplified if the general formula be transformed into a series by expanding the several functions in powers of k. To develop the formula (m) in ascending powers of k, it may be first conveniently put down as follows:—

$$p = \frac{2}{3} - \frac{47}{300}\left(k^2 + \frac{1}{k^2}\right) + k^2 \log\frac{1}{k}$$

$$- \frac{1}{5}\left(\frac{1}{k^2} - 6k^2\right) \log \frac{1+\sqrt{(1-4k^2)}}{2} - \frac{k^3\pi}{80}$$

$$- \frac{\sin^{-1} 2k}{40k^3} + \frac{\frac{31}{k^2} - 63 - 64k^2}{150} \sqrt{(1-4k^2)}.$$

Now $\frac{d}{dk} \log \frac{1+\sqrt{(1-4k^2)}}{2}$

$$= -\frac{1}{k}\left\{(1-4k^2)^{-\frac{1}{2}} - 1\right\}$$

$$= -\frac{1}{k}\left(2k^2 + 6k^4 + 20k^6 + 70k^8 \ldots\right)$$

$\therefore \log \frac{1+\sqrt{(1-4k^2)}}{2}$

$$= -\left(k^2 + \frac{3}{2}k^4 + \frac{10}{3}k^6 + \frac{35}{4}k^8 \ldots\right)$$

and $-\left(\frac{1}{k^2} - 6k^2\right) \log \frac{1+\sqrt{(1-4k^2)}}{2}$

$$= 1 + \frac{3}{2}k^2 - \frac{8}{3}k^4 - \frac{1}{4}k^6 \ldots$$

Also $\frac{\sin^{-1} 2k}{k^3} = \frac{1}{k^3}\left\{2k + \frac{1}{2}\cdot\frac{(2k)^3}{3}\right.$

$$+ \frac{1\cdot3}{2\cdot4}\cdot\frac{(2k)^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{(2k)^7}{7}$$

$$\left. + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{(2k)^9}{9} \ldots \right\}$$

$$= \frac{2}{k^2} + \frac{4}{3} + \frac{12}{5}k^2 + \frac{40}{7}k^4 + \frac{140}{9}k^6 \ldots$$

and $\left(\frac{31}{k^2} - 63 - 64k^2\right)\sqrt{(1-4k^2)}$

$$= \left(\frac{31}{k^2} - 63 - 64k^2\right)\left(1 - 2k^2 - 2k^4 - 4k^6 - 10k^8 \ldots\right)$$

$$= \frac{31}{k^2} - 125 + 130k^4 + 70k^6 \ldots$$

$$\therefore p = \frac{2}{3} - \frac{47}{300}\left(k^2 + \frac{1}{k^2}\right) + k^2 \log\frac{1}{k}$$

$$+ \frac{1}{5}\left(1 + \frac{3}{2}k^2 - \frac{8}{3}k^4 - \frac{1}{4}k^6 \ldots\right) - \frac{k^3\pi}{80}$$

$$- \frac{1}{40}\left(\frac{2}{k^2} + \frac{4}{3} + \frac{12}{5}k^2 + \frac{40}{7}k^4 + \frac{140}{9}k^6 \ldots\right)$$

$$+ \frac{1}{150}\left(\frac{31}{k^2} - 125 + 130k^4 + 70k^6 \ldots\right)$$

$$= k^2\left(\log\frac{1}{k} + \frac{1}{12} - \frac{k\pi}{80} + \frac{4}{21}k^2 + \frac{1}{36}k^4 \ldots\right) \ldots (\mu)$$

which will give a result true to at least four places of decimals.

If k be small, rejecting terms inferior to k^2,

$$p = k^2\left(\log\frac{1}{k} + \frac{1}{12}\right)$$

$$= \frac{b^2}{a^2}\left(\log\frac{a}{b} + \frac{1}{12}\right)$$

which, for an elongated rectangle, gives a good approximation.

And when k is very small, the limiting value for a very elongated rectangle is

$$p = k^2 \log\frac{1}{k} = \frac{b^2}{a^2}\log\frac{a}{b}$$

Or, if k be supposed to denote $\frac{a}{b}$, the reciprocal value, and therefore greater than 1, then, when k is a large number, the limiting value is

$$p = \frac{\log k}{k^2}$$

which is a remarkably singular expression.

The logarithms throughout are, of course, understood to be Napierian.

As it may be interesting to exhibit some practical examples, a series of values of the required probability, computed from the strict formula (m), are tabulated below.

$k = \frac{b}{a}$	Prob. of acute Δ (p).	$k = \frac{b}{a}$	Prob. of acute Δ (p).
·0	·00000	·8	·26597
·1	·02377	·9	·27280
·2	·06769	1·0	·27479
·3	·11636	1·1	·27316
·4	·16243	1·2	·26887
·5	·20163	1·3	·26268
·6	·23181	1·4	·25516
·7	·25284	1·5	·24678

It has been shown that the formula (n) is rigorous down to $k = {\cdot}5$, below which the three final terms retained in the formula (m), insinuating themselves, begin to exhibit a real value, and they assume a considerable magnitude when k becomes small. For example, when $k = {\cdot}1$, the formula (n) gives the negative value $-8{\cdot}22451$, and the value of the three additional or supplementary terms alluded to is $+8{\cdot}24828$, making $p = {\cdot}02377$, as shown in the Table.

1273 (Proposed by the Editor.)—In a given triangle let *three* triangles be inscribed, by joining the points of contact of the inscribed

circle, the points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the *corresponding* sides of these three triangles pass through the same point; also the triangle formed by the three points of intersection will be a *circumscribed copolar* to the original triangle, and the *pole* will be on the *straight line* in which the sides of the given triangle meet the bisectors of its exterior angles.

Solution by Professor CAYLEY.

The theorem is in fact included in the following more general

THEOREM. Let the points O, O′, O″, . . . lie on a conic circumscribed about a triangle ABC; then *first* the polars of the points O, O′, O″, . . . in regard to the triangle (see Note at the end of the Solution) pass through a fixed point Ω. And secondly, if by means of the point O, joining it with the vertices A, B, C, and taking the intersections of these lines with the sides BC, CA, AB, respectively, we form a triangle inscribed in the triangle ABC; and the like for the points O′, O″, . . .; the corresponding sides of the inscribed triangles meet in three points forming a triangle circumscribed about the original triangle ABC, and such that the lines joining the corresponding vertices of the last-mentioned two triangles meet in the point Ω.

But in order to see that the proposed theorem 1273 is in fact included under the foregoing more general one, it is necessary to state the following

SUBSIDIARY THEOREM. Consider a conic inscribed in the triangle ABC, and passing through the points I, J.

Take O the pole of the line IJ in regard to the conic; O′ the point of intersection of the lines joining the vertices of the triangle with the points of contact on the opposite sides respectively; O″ the point of intersection of the lines A*l*, B*m*, C*n*, if *l* be a point on BC such that the lines *l*A, *l*BC, *l*I, *l*J form a harmonic pencil, and the like for the points *m* and *n* respectively.

Then the points O, O′, O″ lie on a conic circumscribed about the triangle ABC.

In fact, if in the subsidiary theorem the inscribed conic be a circle, and the points I, J be the circular points at infinity, the point O will be the centre of the circle, that is, the point of intersection of the interior bisectors of the angles; O′ will be the point of intersection of the lines to the points of contact of the inscribed circle; and O″ the point of intersection of the perpendiculars on the sides of the triangle; and these three points being on a conic circumscribed about the triangle, the general theorem will apply to the three points in question.

I first prove the subsidiary theorem. Taking $x=0$, $y=0$, $z=0$ for the equations of the sides of the triangle, and (α, β, γ), $(\alpha', \beta', \gamma')$ for the coordinates of the points I, J respectively; the equation of the inscribed conic is

$$\begin{vmatrix} \sqrt{x}, & \sqrt{y}, & \sqrt{z} \\ \sqrt{\alpha}, & \sqrt{\beta}, & \sqrt{\gamma} \\ \sqrt{\alpha'}, & \sqrt{\beta'}, & \sqrt{\gamma'} \end{vmatrix} = 0$$

or say,

$$a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0,$$

where

$$a=\sqrt{(\beta\gamma')}-\sqrt{(\beta'\gamma)}=p-p_1 \quad \text{suppose}$$
$$b=\sqrt{(\gamma\alpha')}-\sqrt{(\gamma'\alpha)}=q-q_1 \quad ,,$$
$$c=\sqrt{(\alpha\beta')}-\sqrt{(\alpha'\beta)}=r-r_1 \quad ,,$$

The coordinates of the point of intersection of the lines from the vertices to the points of contact on the opposite sides are

$$x:y:z=\frac{1}{a^2}:\frac{1}{b^2}:\frac{1}{c^2},$$

$$\text{that is, } =\frac{1}{(p-p_1)^2}:\frac{1}{(q-q_1)^2}:\frac{1}{(r-r_1)^2}.$$

The equation of the line IJ is

$$(\beta\gamma'-\beta'\gamma)\,x+(\gamma\alpha'-\gamma'\alpha)y+(\alpha\beta'-\alpha'\beta)\,z=0;$$

or what is the same thing,

$$(p^2-p_1^2)\,x+(q^2-q_1^2)y+(r^2-r_1^2)\,z=0.$$

Representing this for a moment by $\lambda x+\mu y+\nu z=0$, the coordinates of the pole of this line in regard to the inscribed conic $a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0$ are as

$$c^2\mu+b^2\nu : a^2\nu+c^2\lambda : b^2\lambda+a^2\mu.$$

Now $c^2\mu+b^2\nu$

$$\begin{aligned} &=(r-r_1)^2(q^2-q_1^2)+(q-q_1)^2(r^2-r_1^2)\\ &=(r-r_1)(q-q_1)\,[(r-r_1)(q+q_1)\\ &\qquad +(q-q_1)(r+r_1)]\\ &=2(r-r_1)(q-q_1)(qr-q_1r_1);\end{aligned}$$

but observing that $pqr=p_1q_1r_1$ we have

$$qr-q_1r_1=\left(\frac{p_1}{p}-1\right)q_1r_1=-\frac{(p-p_1)\,q_1r_1}{p},$$

$$=-\frac{(p-p_1)\,p_1q_1r_1}{pp_1};\ \text{hence}$$

$$c^2\mu+b^2\nu=-\frac{2\,(p-p_1)(q-q_1)(r-r_1)\,p_1q_1r_1}{pp_1},$$

and the like values for $a^2\nu+c^2\lambda$ and $b^2\lambda+a^2\mu$ respectively; hence, omitting the symmetrical factor, we have for the coordinates of the point in question,

$$x:y:z=\frac{1}{pp_1}:\frac{1}{qq_1}:\frac{1}{rr_1}.$$

Taking the equation of the line A*l* to be $Qy+Rz=0$, those of the lines *l*I, *l*J will be

$$x=\lambda\,(Qy+Rz),\quad x=\lambda'\,(Qy+Rz)$$

$$\text{where } \lambda=\frac{\alpha}{Q\beta+R\gamma},\quad \lambda'=\frac{\alpha'}{Q\beta'+R\gamma'};$$

and the harmonic condition gives $\lambda+\lambda'=0$, that is

$$Q\,(\alpha\beta'+\alpha'\beta)+R\,(\alpha\gamma'+\alpha'\gamma)=0;$$

the equation of the line A*l* is thus found to be

$$(\gamma\alpha'+\gamma'\alpha)\,y=(\alpha\beta+\alpha'\beta)\,x;$$

and since we have the like forms for the equations of the lines B*m* and C*n*, we have for the coordinates of the point of intersection of these three lines

$$x : y : z = \frac{1}{\beta\gamma' + \beta'\gamma} : \frac{1}{\gamma\alpha' + \gamma'\alpha} : \frac{1}{\alpha\beta' + \alpha'\beta},$$

$$\text{that is, } = \frac{1}{p^2 + p_1^2} : \frac{1}{q^2 + q_1^2} : \frac{1}{r^2 + r_1^2}.$$

The equation of a conic circumscribed about the triangle ABC is

$$\frac{\lambda}{x} + \frac{\mu}{y} + \frac{\nu}{z} = 0$$

where λ, μ, ν are arbitrary coefficients; and the condition for the three points being in the conic is thus found to be

$$\begin{vmatrix} (p-p_1)^2, & (q-q_1)^2, & (r-r_1)^2 \\ pp_1, & qq_1, & rr_1 \\ p^2+p_1^2, & q^2+q_1^2, & r^2+r_1^2 \end{vmatrix} = 0$$

but in virtue of the relations

$$(p-p_1)^2 = -2pp + (p^2+p_1^2), \text{ \&c.},$$

this equation is identically true, and the subsidiary theorem is thus proved.

Passing now to the general theorem, I prove the first part of it as follows :—

The equation of a conic circumscribed about the triangle $x=0$, $y=0$, $z=0$ is

$$\frac{A}{x} + \frac{B}{y} + \frac{C}{z} = 0,$$

hence if (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ are the coordinates of any three points on the conic, we have

$$\frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} = 0$$

$$\frac{A}{\alpha'} + \frac{B}{\beta'} + \frac{C}{\gamma'} = 0$$

$$\frac{A}{\alpha''} + \frac{B}{\beta''} + \frac{C}{\gamma''} = 0$$

and thence

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix} = 0.$$

But this is the condition for the intersection of the three lines

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0$$

$$\frac{x}{\alpha'} + \frac{y}{\beta'} + \frac{z}{\gamma'} = 0$$

$$\frac{x}{\alpha''} + \frac{y}{\beta''} + \frac{z}{\gamma''} = 0$$

in a point; and the theorem in question is thus proved. I remark in passing, that the theorem might also be stated as follows :—

The locus of a point O such that its polar in regard to the triangle ABC passes through a fixed point Ω, is a conic circumscribed about the triangle.

To prove the second part of the theorem, take for the coordinates of the points O, O′, O″ respectively (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; then

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix} = 0$$

and if X, Y, Z are the coordinates of the point Ω, then we have

$$\frac{X}{\alpha} + \frac{Y}{\beta} + \frac{Z}{\gamma} = 0$$

$$\frac{X}{\alpha'} + \frac{Y}{\beta'} + \frac{Z}{\gamma'} = 0$$

$$\frac{X}{\alpha''} + \frac{Y}{\beta''} + \frac{Z}{\gamma''} = 0.$$

The equations of the sides of the inscribed triangle obtained by means of the point O are

$$-\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0$$

$$\frac{x}{\alpha} - \frac{y}{\beta} + \frac{z}{\gamma} = 0$$

$$\frac{x}{\alpha} + \frac{y}{\beta} - \frac{z}{\gamma} = 0$$

and the like for the triangles obtained by means of the points O′ and O″ respectively. Hence for a set of corresponding sides of the three triangles, we have, *e.g.*,

$$-\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0$$

$$-\frac{x}{\alpha'} + \frac{y}{\beta'} + \frac{z}{\gamma'} = 0$$

$$-\frac{x}{\alpha''} + \frac{y}{\beta''} + \frac{z}{\gamma''} = 0,$$

and it is clear that these equations are simultaneously satisfied by the values

$$x : y : z = -X : Y : Z.$$

And the like for the other sets of corresponding sides, that is, we have for the coordinates of the vertices of the resulting triangle

$$(-X : Y : Z),$$
$$(X : -Y : Z),$$
$$(X : Y : -Z);$$

and thence also the equations of the sides of the triangle in question are

$$\frac{y}{Y} + \frac{z}{Z} = 0, \quad \frac{z}{Z} + \frac{x}{X} = 0, \quad \frac{x}{X} + \frac{y}{Y} = 0,$$

that is, it is a triangle circumscribed about the triangle ABC. The equations of the lines joining the corresponding vertices of the two triangles are

$$\frac{y}{Y}=\frac{z}{Z},\ \frac{z}{Z}=\frac{x}{X},\ \frac{x}{X}=\frac{y}{Y},$$

and these lines meet in the point (X : Y : Z) which is the point Ω, the intersection of the polars of O, O′, O″; the demonstration of the theorem is thus completed.

NOTE.—The expression Polar of a point in regard to a triangle denotes a line constructed as follows: viz., O being the point and ABC the triangle, then taking on BC a point a, the harmonic in regard to the points B and C of the intersection of BC by AO; and in like manner on CA and AB the points b and c respectively, the three points a, b, c lie on a line which is the polar of the point O. If the equations of the sides are $x=0$, $y=0$, $z=0$, and the coordinates of the point are (α, β, γ), then the equation of the polar is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0;$$

the equation may also be written

$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)^2 xyz=0,$$

and it thus appears that the line just defined as the polar is in fact the second or line polar of the point in regard to the three lines BC, CA, AB considered as forming a cubic curve.

1416 (Proposed by Professor SYLVESTER.) Show that the area of the perspective representation, in a given picture, of a triangle of given area in a fixed plane, varies as the product of the distances of the angles of the perspective representation from the vanishing line.

Solution by Mr. W. K. CLIFFORD.

Let ABC be the perspective representation of the triangle, DE the vanishing line. Let BC meet DE in D, and join AD. If abc is the triangle represented, AD is the picture of a line through a parallel to bc. If therefore B and C are fixed, the point A can only move along AD. But the area ABC varies as the perpendicular from A on BC, which is in a constant ratio to the perpendicular from A on DE, because A lies on a fixed line through the intersection of BC and DE. Since then when two of the perpendiculars on the vanishing line are fixed, the area varies directly as the remaining one; therefore when all vary, the area varies as the product of the three.

1441 (Proposed by Dr. SALMON, F.R.S.)—A pair of dice is thrown, or a teetotum of n sides is spun an indefinite number of times, and the numbers turning up are added together (as in the game of Steeplechase); what is the chance (or rather the limit of the chance) that a given high number will be actually arrived at? For instance, if the game was won by whoever first got 100, and that getting 101 or 102 would not do.

General Solution by W. S. B. WOOLHOUSE.

The Proposer's Solution, given on p. 41, is only an approximation or limit; but it is abundantly correct for high numbers, in accordance with the restriction originally made in the Question. It is here intended to determine generally the exact chance for any number.

Taking the proposition of a teetotum of n sides, let p_x denote the probability of arriving precisely at a number x. Then the number which immediately precedes x must be either $x-1$, $x-2$, $x-3$, . . ., or $x-n$; and on the realization of any one of these numbers, the chance that x will be the next following number is evidently $\frac{1}{n}$;

$$\therefore\ p_x=\frac{1}{n}\left(p_{x-1}+p_{x-2}\cdots\cdots+p_{x-n}\right)\cdots(1),$$

that is, the probability with respect to any number is the average of the probabilities of the n numbers which immediately precede it, and this must obviously approximate towards a constant limit for high numbers, as determined by Dr. Salmon.

Putting $x-1$ for x,

$$p_{x-1}=\frac{1}{n}\left(p_{x-2}+p_{x-3}\cdots+p_{x-n}+p_{x-n-1}\right)$$

$$\therefore\ p_x-p_{x-1}=\frac{1}{n}\left(p_{x-1}-p_{x-n-1}\right)$$

$$\therefore\ p_x=p_{x-1}+\frac{1}{n}\left(p_{x-1}-p_{x-n-1}\right)$$

$$=\frac{n+1}{n}p_{x-1}-\frac{1}{n}p_{x-n-1}$$

$$=hp_{x-1}-kp_{x-n-1}\cdots\cdots\cdots\cdots(2),$$

by which successive values may be easily calculated.

Let $p_x=h^x q_x$; then the last equation becomes

$$h^x q_x=h\left(h^{x-1}q_{x-1}\right)-k\left(h^{x-n-1}q_{x-n-1}\right)$$

$$\therefore\ q_x=q_{x-1}-kh^{-n-1}q_{x-n-1}$$

$$=q_{x-1}-\lambda q_{x-n-1}\cdots\cdots\cdots\cdots\cdots(3).$$

This equation also determines successive values of q *ad libitum*, provided that $n+1$ consecutive values are given, and its general solution, as a functional equation, will therefore contain $n+1$ arbitrary constants.

Now, if $A_x=x$,

$$B_x=\frac{x(x-1)}{2},$$

$$C_x=\frac{x(x-1)(x-2)}{2\cdot 3},$$

&c &c.

then $A_x - A_{x-1} = 1,$

$$B_x - B_{x-1} = A_{x-1},$$
$$C_x - C_{x-1} = B_{x-1},$$
&c. &c.;

and particular solutions of the equation (3) will be

$$f(x) = 1_x - \lambda A_{x-n} + \lambda^2 B_{x-2n} - \lambda^3 C_{x-3n} + \&c.$$
$$f(x-1) = 1_{x-1} - \lambda A_{x-n-1} + \lambda^2 B_{x-2n-1} - \lambda^3 C_{x-3n-1} + \&c.$$
&c. &c.

Here $f(x) = 1$ from $x = 0$ to n,
$f(x-1) = 1$ from $x = 1$ to $n+1$,
&c. &c.,
and $f(x) = 0$ for negative values of x.

Therefore, as the equation (3) is linear, the general solution for positive values of x is

$$q_x = q_0 f(x) + (q_1 - q_0) f(x-1) + (q_2 - q_1) f(x-2) \ldots\ldots + (q_n - q_{n-1}) f(x-n) \ldots\ldots\ldots (4),$$

for when $x = 0, 1, 2, \ldots\ldots n$, this becomes $q_0, q_1, q_2, \ldots q_n$ respectively. To find the last mentioned values of q, we have, by the nature of the question, $p_0 = 1$, $p_1 = \frac{1}{n} = k$; and as the conventional values which immediately precede these must be regarded as zeros, the equation (1) or (2) evidently gives $p_2, p_3, \ldots .p_n$ respectively $= kh, kh^2, \ldots\ldots kh^{n-1}$; $\therefore q_0 = 1$, and $q_1, q_2, \ldots\ldots q_n$ each $= kh^{-1} = \frac{1}{n+1}$. Hence, under the conditions of the question, the general solution (4) becomes

$$q_x = q_0 f(x) + (q_1 - q_0) f(x-1)$$
$$= (1 - \lambda A_{x-n} + \lambda^2 B_{x-2n} - \lambda^3 C_{x-3n} + \&c.) - \frac{n}{n+1}(1 - \lambda A_{x-n-1} + \lambda^2 B_{x-2n-1} - \lambda^3 C_{x-3n-1} + \&c.)$$
$$= \frac{1}{n+1}\left\{1 - x\lambda + \frac{x(x-2n-1)}{2}\lambda^2 - \frac{x(x-3n-1)(x-3n-2)}{2.3}\lambda^3 + \frac{x(x-4n-1)(x-4n-2)(x-4n-3)}{2.3.4}\lambda^4 - \&c.\right\}$$

$$\therefore p_x = \frac{1}{n}\left\{h^{x-1} - \frac{x}{n+1}h^{x-n-1} + \frac{x(x-2n-1)}{2(n+1)^2}h^{x-2n-1} - \frac{x(x-3n-1)(x-3n-2)}{2.3(n+1)^3}h^{x-3n-1} + \frac{x(x-4n-1)(x-4n-2)(x-4n-3)}{2.3.4(n+1)^4}h^{x-4n-1} - \&c.\right\} \ldots\ldots\ldots\ldots\ldots (5).$$

where $h = \frac{n+1}{n}$, the number of terms of the series after the first being the integer quotient of $\frac{x}{n+1}$.

Examples.—Take the case of a single die, or a teetotum of six sides, that is $n = 6$, and let $x = 20$; then

$$p = \frac{1}{6}\left\{\left(\frac{7}{6}\right)^{19} - \frac{20}{7}\left(\frac{7}{6}\right)^{12} + \frac{10}{7}\left(\frac{7}{6}\right)^{7}\right\} = \cdot 28562.$$

When $x = 6$, $p = \frac{1}{6}\left(\frac{7}{6}\right)^5 = \cdot 36023$, which is a higher probability than that of any other number.

Other maxima are

$x = 11$, $p = \cdot 29389$,
$x = 16$, $p = \cdot 28707$,
$x = 21$, $p = \cdot 28597$,
$x = 27$, $p = \cdot 28577$,
$x = 32$, $p = \cdot 28573$,
$x = 38$, $p = \cdot 28572$,
&c. &c.

According to the question, a trial commences from zero, and is not completed until the success or failure is determined by the proposed number being either exactly reached or passed over. A trial may be made to begin with 1, or 2, or 3, &c. Thus, if the number to be accumulated be 20, and a trial begin from 1, the numbers thrown must make exactly 19; if a trial begin from 2, they must make exactly 18, and so on. A little consideration will show that if there be

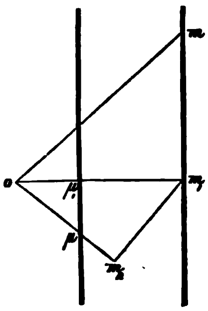
6	trials	commencing	from	zero
5	,,	,,	,,	1
4	,,	,,	,,	2
3	,,	,,	,,	3
2	,,	,,	,,	4
1	,,	,,	,,	5

the expectation will then *accurately* be a constant quantity (6) for *all* numbers; and, as there are 21 trials, the average chance on one trial will be

$$\frac{6}{21} \text{ or } \frac{2}{7}.$$

1479 (Proposed by Mr. W. K. Clifford.)—Prove that the ordinary inverse of the *Tangential inverse* is the second positive pedal; and that the *Tangential inverse* of the ordinary inverse is the second negative pedal of the primitive.

Solution by Archer Stanley.



The symbolical solution of this question results at once from Arts. 2—5, of Mr. Clifford's Solution of 1442. The following direct consideration of the objects there symbolized will render their relations still more evident.

Let o be the origin, mm_1 any tangent to the primitive curve (m), m_1 the foot of the perpendicular upon the same, and μ_1 the inverse of m_1 with respect to any fundamental circle around o. The parallel to $m\,m_1$ through μ_1, which by definition envelopes the tangential inverse (μ), is (by construction) the polar of m_1, and therefore has for its envelope the reciprocal of the first positive pedal (m_1); its point of contact μ is of course on the perpendicular om_2 let fall from the origin upon the tangent m_1m_2 to this pedal. The first part of the theorem will be obvious on observing that μ and m_2 are inverse points, and that m_2 is on the second positive pedal. The second part of the theorem will be equally manifest on regarding (m_2) as the primitive, and, consequently, (m) as the second negative pedal.

It will be observed that the radii vectores om, $o\mu$ to corresponding points of tangentially inverse curves are equally inclined to the corresponding and parallel tangents (the angle of inclination in each case being equal to om_1m_2), so that the asymptotes of two such curves always correspond in pairs. The properties of tangential inverse curves, in fact, are precisely correlative to those of the ordinary inverse curves, of which they are always the reciprocals. Thus the inverse of a line being a circle through the origin, and that of any other circle also a circle; the tangential inverse of a point is a parabola with focus at the origin, and that of any other conic, with a focus at the origin, is a conic of the same species, and coincident focus.

In short, the following general relation may be at once deduced from Dr. Hirst's Theorem 1, (Quest. 1471); a demonstration of which, it may be remarked, at once results from the correction of a few slight and obvious errors in Art. 6 of the Solution of Quest. 1442 :—

The primitive being a curve of the nth class, touching the line at infinity α times, and having a focus at the origin of the order β; its tangential inverse is of the $(2n-\alpha-2\beta)^{\text{th}}$ class, touches the line at infinity $(n-2\beta)$ times, and has a focus at the origin of the order $(n-\alpha-\beta)$.

1497 (Proposed by Mr. W. K. Clifford, Trinity College, Cambridge.)—Given three points by equations of the form $lx+my+nz=0$, prove that the area of the triangle contained by them is

$$\frac{\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}}{(l_1+m_1+n_1)\,(l_2+m_2+n_2)\,(l_3+m_3+n_3)}$$

that of the triangle of reference being unity.

Solution by W. A. Whitworth.

We will first prove the corresponding result in triangular coordinates (α, β, γ denoting the ratios of the areas of the triangles PBC, PCA, PAB to the triangle of reference ABC, and the relation $\alpha+\beta+\gamma=1$ being therefore always true). We shall then deduce Mr. Clifford's result as a corollary.

We purpose avoiding all actual multiplication and division by retaining the determinants throughout. This plan, though it occupies much more space than the other, reduces the labour of the proof to a minimum.

Let P_1, P_2, P_3 be three points whose triangular coordinates referred to the triangle ABC are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$. The area of the triangle formed by joining them will be

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Let AP_1 be joined and produced to meet BC in Q and P_2P_3 in R; then, putting (α, β, γ) for the coordinates of R, we have

$$\frac{\Delta P_1P_2P_3}{\Delta AP_2P_3} = \frac{P_1R}{AR} = \frac{\alpha_1-\alpha}{1-\alpha} = 1-\frac{1-\alpha_1}{1-\alpha}.$$

But since $(\alpha\beta\gamma)$ lies on the lines AP_1 and P_2P_3, α, β, γ are determined by the equations

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0, \quad \frac{\beta}{\beta_1} = \frac{\gamma}{\gamma_1} = \frac{1-\alpha}{1-\alpha_1};$$

hence, eliminating β and γ, we get

$$\begin{vmatrix} (1-\alpha_1)\alpha & \beta_1(1-\alpha) & \gamma_1(1-\alpha) \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0,$$

$$\text{or} \quad \begin{vmatrix} \frac{1-\alpha_1}{1-\alpha} & \beta & \gamma \\ 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \end{vmatrix} = 0,$$

$$\text{or} \quad \begin{vmatrix} 1-\frac{1-\alpha_1}{1-\alpha} & 0 & 0 \\ 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} 1 & \beta_1 & \gamma_1 \\ 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$\therefore 1 - \frac{1-a_1}{1-a} = \frac{\Delta P_1P_2P_3}{\Delta AP_2P_3} = \frac{\begin{vmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}} \quad (1).$$

Applying this result to the comparison of the triangles AP_2P_3, ABP_3, and ABC, we get

$$\frac{\Delta AP_2P_3}{\Delta ABP_3} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}} \quad \ldots\ldots\ldots\ldots(2),$$

$$\frac{\Delta ABP_3}{\Delta ABC} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} \quad \ldots\ldots\ldots\ldots(3).$$

Compounding (1), (2), and (3), and observing that the area of the triangle ABC is unity, we obtain

$$\Delta P_1P_2P_3 = \begin{vmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots(4).$$

COR. I.—If the points P_1, P_2, P_3 be given by their tangential equations of the form

$$lx + my + nz = 0,$$

then, since l, m, n are proportional to the triangular areas a, β, γ, we shall have to substitute in our result the values given by

$$\frac{a}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{1}{l+m+n}$$

(since $a+\beta+\gamma=1$).

Hence our expression will become

$$\frac{\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}}{(l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3)}$$

which is the form given by Mr. Clifford.

COR. II.—If the points be determined by ordinary trilinear instead of triangular coordinates, our result must be modified by replacing a, β, γ by quantities proportional to aa, $b\beta$, $c\gamma$; and we shall thus get

$$\frac{\Delta P_1P_2P_3}{\Delta ABC} = \frac{\begin{vmatrix} aa_1 & b\beta_1 & c\gamma_1 \\ aa_2 & b\beta_2 & c\gamma_2 \\ aa_3 & b\beta_3 & c\gamma_3 \end{vmatrix}}{\begin{vmatrix} 2\Delta & 0 & 0 \\ 0 & 2\Delta & 0 \\ 0 & 0 & 2\Delta \end{vmatrix}}$$

where Δ denotes the area of the triangle ABC;

$$\therefore \quad \Delta P_1P_2P_3 = \frac{abc}{8\Delta^2}\begin{vmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

1498 (Proposed by W. A. WHITWORTH.)—If c_1, c_2, c_3, c_4 .... c_n be the coefficients of x, x^2, x^3, x^4 .... x^n when $(1+x)^n$ is expanded by the Binomial Theorem, prove that

$$1.c_1 + 2.c_2 + 3.c_3 + \ldots + n.c_n = 2^{n-1}.n,$$

$$1^2.c_1 + 2^2.c_2 + 3^2.c_3 + \ldots + n^2.c_n = 2^{n-2}(n^2+n),$$

$$1^3.c_1 + 2^3.c_2 + 3^3.c_3 + \ldots + n^3.c_n = 2^{n-3}(n^3+3n^2),$$

$$1^4.c_1 + 2^4.c_2 + 3^4.c_3 + \ldots + n^4.c_n = 2^{n-4}(n^4+6n^3+3n^2-2n),$$

$$1^5.c_1 + 2^5.c_2 + 3^5.c_3 + \ldots + n^5.c_n = 2^{n-5}(n^5+10n^4+15n^3-10n^2);$$

and find an expression for the sum of the series

$$1^r.c_1 + 2^r.c_2 + 3^r.c_3 + \ldots + n^r.c_n.$$

Solution by F. D. THOMSON, B.A.

The series

$$n.1^r + \frac{n(n-1)}{1.2}2^r + \&c. + n^r$$

is obtained from the series

$$n(x+1)^r + \frac{n(n-1)}{1.2}(x+2)^r + \ldots + (x+n)^r$$

by putting $x=0$.

Now this last series may be written

$$nDx^r + \frac{n(n-1)}{1.2}D^2x^r + \ldots + D^nx^r,$$

with the usual notation of Finite Differences.

Separating symbols, the series becomes

$$\left\{(1+D)^n-1\right\}x^r, \text{ or } \left\{(2+\Delta)^n-1\right\}x^r.$$

The general expression for the proposed series is therefore

$$\left\{(2+\Delta)^n-1\right\}0^r, \text{ or } (2+\Delta)^n 0^r \dots\dots\dots (a),$$

which, when expanded, becomes

$$2^n\left\{n\frac{\Delta 0^r}{2}+\frac{n(n-1)}{1.2}\cdot\frac{\Delta^2 0^r}{2^2}\right.$$

$$\left.+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{\Delta^3 0^r}{2^3}-\&c.\right\}.$$

Hence $1^r.c_1+2^r.c_2+3^r.c_3+\dots+n^r c_n$

$$=2^{n-r}\left\{2^{r-1}c_1.\Delta 0^r+2^{r-2}c_2.\Delta^2 0^r+\dots c_r.\Delta^r 0^r\right\}$$

For example, let $r=5$, then $\Delta 0^5=1$, $\Delta^2 0^5=30$) $\Delta^3 0^5=150$, $\Delta^4 0^5=240$, $\Delta^5 0^5=120$; and the series becomes

$$2^{n-5}n\left\{2^4+\frac{n-1}{2}.2^3.30+\frac{(n-1)(n-2)}{6}.2^2.150\right.$$

$$+\frac{(n-1)(n-2)(n-3)}{24}.2.240$$

$$\left.+\frac{(n-1)(n-2)(n-3)(n-4)}{120}.120\right\}$$

$$=2^{n-5}\left\{n^5+10n^4+15n^3-10n^2\right\}.$$

ELLIPSE AND HYPERBOLA.

NOTE by W. S. B. WOOLHOUSE.

In the May number of the *Educational Times*, (see page 70 of this volume) Professor De Morgan has taken up the subject of the mutual relationship between the ellipse and hyperbola; and by ingeniously designating the circle as the *equilateral ellipse*, he has facetiously suggested that, with respect to the ellipse and hyperbola, the various geometrical properties which involve the circle, or equilateral ellipse, might be considered as a robbery of the equilateral hyperbola, unless the latter were allowed, somehow or other, to make reprisals. As a case in illustration of this curious suggestion, Professor De Morgan instances the well known theorem that the circle on the major (transverse) axis of an ellipse or hyperbola is the locus of the intersection of a tangent to *either curve* with a perpendicular from a focus of the same curve. There can be no doubt whatever that every property involving the circle has its analogue, real or imaginary, involving the equilateral hyperbola. Indeed, it will be found that in general the hyperbola has really the advantage; since, under a conjugate species, it admits of being introduced in another form, equally tangible; and for this special appropriation on the part of the hyperbola, the ellipse can only feign to retaliate on the unreal pretence of possessing two additional foci on the minor axis.

The analogues of any property, when its conditions and relations are stated analytically, may be obtained, without renewing the investigation, by simply connecting the factor $\sqrt{(-1)}$, or i, with such symbols as shall preserve the essential equations in their requisite form, and shall at the same time, if possible, entirely eliminate the imaginary character. Thus, taking Professor De Morgan's example alluded to, the primitive theorem, as appertaining to the ellipse, is the following

THEOREM.—*The locus of the intersection of a tangent to an ellipse with the perpendicular from a focus is the circle on the major axis.*

The essential equations which determine this theorem are

Ellipse .... $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Tangent .... $\dfrac{x\xi}{a^2}+\dfrac{y\eta}{b^2}=1$

Perpendicular on tangent .... $\dfrac{a^2\xi}{x}-\dfrac{b^2\eta}{y}=\dfrac{a^2}{x}\sqrt{(a^2-b^2)}$

Locus or Circle .... $\xi^2+\eta^2=a^2$

the last equation being the result of the elimination of x and y from the other three.

By substituting ib for b, the equations become

Hyperbola .... $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Tangent .... $\dfrac{x\xi}{a^2}-\dfrac{y\eta}{b^2}=1$

Perpendicular .... $\dfrac{a^2\xi}{x}+\dfrac{b^2\eta}{y}=\dfrac{a^2}{x}\sqrt{(a^2+b^2)}$

Locus or Circle.... $\xi^2+\eta^2=a^2$.

Here the ellipse becomes the hyperbola, the perpendicular is still drawn from the focus to the tangent, and the resulting locus is unchanged.

By interchanging the symbols ab, and also xy, in the latter set of equations, they will show that the locus of the intersection of a tangent to the conjugate hyperbola $\left(\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=1\right)$ with the perpendicular from one of its foci is the circle described on the *minor* axis. The distance of each of the foci from the centre is $\sqrt{(a^2-b^2)}$ for the ellipse, and $\sqrt{(a^2+b^2)}$ for the hyperbola; and when ab are interchanged this becomes imaginary for the ellipse, but remains unaltered for the hyperbola. Thus, it will appear generally that theorems which appertain to the hyperbola will, when ab are interchanged, be geometrically true with the conjugate hyperbola; but not so with the ellipse. For, although analytically true, it would avail but little, geometrically, to assert that the locus of the intersection of a tangent to an ellipse with the perpendicular from one of its imaginary foci, is the circle on the minor axis.

It may be as well to premise that the interchanges here effected are to be regarded as purely symbolic, and not geometrical.

To transform the locus into an equilateral hyperbola, $i\eta$ must be substituted for η; then, that the tangent and perpendicular may not become imaginary, iy must be put in place of y; and finally,

if in the first set of equations it be required to retain the ellipse as the primary curve, ib must take the place of b. The ellipse and tangent are thus unchanged, but the perpendicular becomes

$$\frac{a^2\xi}{x} + \frac{b^2\eta}{y} = \frac{a^2}{x}\sqrt{(a^2+b^2)},$$

which is Professor De Morgan's supplemental line, and is, in fact, drawn through a focus of the hyperbola perpendicular to the tangent to the ellipse at the point $(x, -y)$ or $(-x, y)$. Whether the ellipse or hyperbola be taken as the primary curve, let the other be denominated the companion curve; and it will hence be found that if a pair of tangents be drawn to the primary curve, mutually intersecting in either axis, and a perpendicular from a focus of the companion curve be drawn to either of these tangents and produced to intersect the other, the locus of the point of intersection will be the equilateral hyperbola. This is Professor De Morgan's new geometrical property under a somewhat different enunciation. When analytically considered, we have seen that it is perfectly analogous to the theorem first stated, but the analogy in point of geometrical construction is not by any means so obvious. Such an analogy indeed does not necessarily follow, since simple changes of algebraic sign may be sufficiently potent to complicate all geometrical conformity. This would seem to be especially the case with the neat and beautiful theorem last quoted by Professor De Morgan, that if two equal ellipses or hyperbolas be placed vertex to vertex, and one roll upon the other, each rolling focus describes a circle about a fixed focus. The analogous property, in which the loci are equilateral hyperbolas concentric with the companion foci, is readily inferred from what precedes, but it does not present any geometrical elegance. Fortunately, however, the original theorem possesses enough for them both.

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